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B.S. Sashikiran

x Sin nx + n Cos nx Cos nx

x. Cos nx - Sin nx Sin nx)

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CALCULUS

PART I
DIFFERENTIAL CALCULUS

PART II
INTEGRAL CALCULUS
AND
DIFFERENTIAL EQUATIONS

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If, to each value of x, there correspond more than one value of y, then y is called a *many* or *multiple valued* function of x;

$$e.g., \quad \text{if } y^2 = a^2 - x^2, \quad y = \pm\sqrt{a^2 - x^2}.$$

To each value of x, y has two values equal in magnitude and opposite in sign. Therefore y is a two valued function of x.

If $y^3 - 6y^2 + 11y = x$, *three* values of y correspond to each value of x and these values are got by solving this equation of the third degree. For example, if $x = 6$, we have $y^3 - 6y^2 + 11y - 6 = 0$. This gives $y = 1$, 2, or 3. Therefore y is a three valued function of x.

(*b*) *Explicit and implicit functions.*

If the functional relation between two variables x and y is expressed in the form $y = f(x)$, y is called an *explicit function* of x;

$$e.g., \quad y = x^3 + 2x - 3, \quad y = \sin^2 x - \cos x.$$

In the case of explicit functions, the value of the dependent variable for any given value of the independent variable can be obtained by direct substitution in the functional relation.

If the relation between two variables x and y is expressed in the form $f(x, y) = 0$, we may consider y as a function of x or x as a function of y. In such cases, the dependent variable is an *implicit function* of the independent variable. In the case of an implicit function, the value or values of the dependent variable for the given value or values of the independent variable cannot be obtained by direct substitution.

e.g., (1) $x^3 + y^3 = 3axy$.
(2) $\cos x + ay = b \tan y$.
(3) $y = e^{xy}$.

(*c*) *Odd and even functions.*

When there is no change in the sign of $f(x)$ when x is changed to $-x$, that function is called an *even function*.

e.g., $y = \cos x$; $y = 4x^4 - 3x^2 + 6$; $y = e^x + e^{-x}$. If $f(x)$ is an even function, then $f(x) = f(-x)$. If the sign of $f(x)$ is changed when x is changed to $-x$, then it is called an *odd function*. So, if $f(x)$ is odd, we get $f(x) = -f(-x)$;

$$e.g., \quad y = \sin x, \quad y = x^3 + 6x, \quad y = \tan x.$$

There are certain functions which are neither odd nor even;

$$e.g., \quad y = ax^2 + bx + c, \quad y = 3\sin x + 4\cos x.$$

(*d*) *Inverse functions.*

From every function $y = f(x)$, we may be able to deduce what is known as the *inverse function* by expressing the independent variable in terms of the dependent variable.

For example, (1) if $y = \dfrac{x+1}{x-1}$, x can be expressed as a function of y;

$$x = \frac{y+1}{y-1}.$$

(2) If $y = \sin x$, $x = \sin^{-1} y$.

(3) If $y = a^x$, $x = \log_a y$.

$\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ are also called *arc* sin x, *arc* cos x, *arc* tan x respectively.

(*e*) *Algebraic and transcendental functions.*

A polynomial in x is a function of the form

$$a_0 x^m + a_1 x^{m-1} + \dots + a_m,$$

where $a_0, a_1, \dots a_m$ are constants and m is positive. If a fraction has numerator and denominator both polynomials, it is said to be a *rational function* of x.

e.g., $R(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

e.g., $\dfrac{x^3(2x-7)}{(x-1)^2(3x+4)}$.

If y can be expressed by an equation of the form

$$Py^n + Qy^{n-1} + \dots = 0,$$

where P and Q are rational integral functions of x, then y is said to be an *algebraic function* of x.

For example, $y^4 - (3y^2 + 3y + 1)x = 0$

$$y^2 - 2y - x^2 = 0.$$

All functions of x which are not rational or algebraic are called *transcendental functions*;

e.g., $\sin x$, $\log x$, e^x, $\tan^{-1} x$.

Circular functions are the sine and cosine functions of elementary trigonometry and their inverses and the functions derived from them.

The other functions—hyperbolic functions, exponential functions, logarithmic functions will be defined later.

Examples.

Ex. 1. If $f(x) = 3x^3 - 5x^2 + 6x - 4$, find the values of $f(1)$, $f(2)$, $f(0)$, $f(-1)$, and $f(-2)$.

$$f(1) = 3(1)^3 - 5(1)^2 + 6(1) - 4 = 0.$$
$$f(2) = 3(2)^3 - 5(2)^2 + 6(2) - 4 = 12.$$
$$f(0) = 3(0)^3 - 5(0)^2 + 6(0) - 4 = -4.$$
$$f(-1) = 3(-1)^3 - 5(-1)^2 + 6(-1) - 4 = -18.$$
$$f(-2) = 3(-2)^3 - 5(-2)^2 + 6(-2) - 4 = -60.$$

Ex. 2. If $f(x) = A\cos x + B\sin x$, show that

$$f(x + 2\pi) = f(x).$$

$$\begin{aligned} f(x+2\pi) &= A\cos(x+2\pi) + B\sin(x+2\pi) \\ &= A\cos x + B\sin x \\ &= f(x). \end{aligned}$$

$f(x)$ is said to periodic with period 2π.

Ex. 3. If $f(x) = \log x$, show that

$$f(abc) = f(a) + f(b) + f(c).$$

$$\begin{aligned} f(a) + f(b) + f(c) &= \log a + \log b + \log c \\ &= \log(abc) \\ &= f(abc). \end{aligned}$$

Ex. 4. If $3x^2 - 7xy + 2y^2 + 2x - y + 3 = 0$, find y when $x = 3$.

If $x = 3$ is substituted in the equation, we get

$$27 - 21y + 2y^2 + 6 - y + 3 = 0$$

i.e., $2y^2 - 22y + 36 = 0$

i.e., $2(y-2)(y-9) = 0.$

$\therefore\ y = 2$ or 9.

Exercises I.

1. If $f(x) = (2x-1)(x-3)$, find the values of $f(0)$, $f(1), f(2), f(\frac{1}{2}), f(3)$.

2. If $f(x) = x^3 + 2x + 1$, find $f(a^2)$.

3. If $f(x) = \tan x$, show that

$$f(x-y) = \frac{f(x) - f(y)}{1 + f(x)f(y)}.$$

4. If $f(x) = x^2 + x - 1$, simplify

$$f(x+1) - 3f(x) + 2f(x-1).$$

5. If $f(x) = x + \frac{1}{x}$, show that $f(x) = f\left(\frac{1}{x}\right)$.

6. If $f(x) = A\sin^4 x + B\cos^2 x + C$, show that

$$f(x) = f(\pi + x) = f(\pi - x).$$

7. If $f(x, y) = ax^2 + 2hxy + by^2$, find $f(y, x), f(x, x)$ and $f(y, y)$.

8. If $y = f(x) = \dfrac{ax+b}{cx-a}$, show that $x = f(y)$.

9. In the following functions, which are odd functions and which are even functions ?—

(1) $ax^7 + bx^5 + cx^3 + dx$.

(2) $ax^6 + bx^4 + cx^2 + d$.

(3) (*a*) $\sin x$, (*b*) $\cos x$, (*c*) $\tan x$, (*d*) $\operatorname{cosec} x$, (*e*) $\sec x$, (*f*) $\cot x$.]

(4) (*a*) $\dfrac{e^x - e^{-x}}{x}$, (*b*) $\dfrac{e^x - e^{-x}}{2}$.

LIMITS

§ 5. Variable tending to a limit.

Consider the geometrical progression

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots$$

If S denotes the sum to n terms of the series,

$$S = 2\left(1 - \frac{1}{2^n}\right) = 2 - \frac{1}{2^{n-1}}.$$

As n becomes very large, $\frac{1}{2^{n-1}}$ approaches the value zero and can be made as small as we please by taking n sufficiently large; hence the sum of n terms of the series can be made as near 2 as we please. Therefore we say that the limit of S as n tends to infinity is 2.

Let x be a real variable which assumes an infinity of values successively according to some definite rule. If the successive values of x approach a definite number a in such a way that the difference $|x - a|$ becomes ultimately less than any positive quantity ϵ, however small, we say that 'x tends to limit a'. We express this by writing $x \to a$ or limit $x = a$.

§ 6. Limit of a function.

In the function $y = f(x)$, we can calculate the values of y corresponding to the values of x. As the values of x approach the value a, the corresponding values of y may approach another value. If that is the case, we say that $f(x)$ approaches a limit. We can define this as follows :

The necessary and sufficient condition for $f(x)$ tending to a limit l is that given any positive number ϵ, however small, we can always find a corresponding positive number δ such that

$$|f(x) - l| < \epsilon \text{ when } |x - a| < \delta.$$

We express this as $\lim\limits_{x \to a} f(x) = l$.

Example.

Find $\underset{x\to 2}{\text{Lt}} \dfrac{x^2-4}{x-2}$.

Let y be $\dfrac{x^2-4}{x-2}$.

Any value other than 2 can be given to x. When x takes the value 2, y is of the indeterminate form 0/0. When x is given the values tending to 2, the corresponding values of y are given in the table below :

x	0	·5	1	1·5	1·6	1·7	1·8	1·9	1·99	1·999	3
y	2	2·5	3	3·5	3·6	3·7	3·8	3·9	3·99	3·999	5

2·5	2·4	2·3	2·2	2·1	2·05	2·04	2·02	2·01	2·099
4·5	4·4	4·3	4·2	4·1	4·05	4·04	4·02	4·01	4·009

As the values of x approach 2, the values of y approach 4. To make the difference between y and 4 as little as possible, it is enough to make the difference between x and 2 correspondingly very little.

$$\text{Hence } \underset{x\to 2}{\text{Lt}} \frac{x^2-4}{x-2} = 4.$$

Another method. From the definition of limits, it is not easy to determine the limits of functions. In practice, the following method is adopted :—

Put $x = 2 + h$ in the function $\dfrac{x^2-4}{x-2}$.

$$\therefore \quad \frac{x^2-4}{x-2} = \frac{(2+h)^2-4}{(2+h)-2} = 4 + h.$$

As x tends to 2, h tends to zero.

$$\therefore \quad \underset{x\to 2}{\text{Lt}} \frac{x^2-4}{x-2} = \underset{h\to 0}{\text{Lt}} (4+h) = 4.$$

§ 7. Limit and value of a function.

The value of x^2+3x-2 when $x=1$ is 2 and is obtained by directly substituting 1 for x in the function. In this case the value of the function when $x=1$ happens to be the same as the limit when x tends to one. It must however be noted that the value and the limit are obtained by entirely different processes. In finding the limit, note that x is not allowed to take the value 1.

§ 8. Rules for finding the limit of a function.

1. If we require limit $f(x)$ (when it is a rational function) when $x \to a$, we put $x=a+h$. Expanding the numerator and the denominator of the function, we will find that h or some power of h is common factor of the numerator and the denominator. Cancelling this common factor, we make $h \to 0$ and get the limit.

Consider the limit of $y=\dfrac{x^2-a^2}{x-a}$ when $x \to a$.

Put $x=a+h$. When $x \to a$, $h \to 0$.

$$\frac{x^2-a^2}{x-a}=\frac{(a+h)^2-a^2}{(a+h)-a}=\frac{2ah+h^2}{h}=2a+h.$$

$$\therefore \quad \underset{x\to a}{\mathrm{Lt}}\frac{x^2-a^2}{x-a}=\underset{x\to 0}{\mathrm{Lt}}(2a+h)=2a.$$

2. If we require limit $f(x)$ when $x \to 0$, the substitution as in the previous case reduces to $x=h$ which is evidently unnecessary. In such cases, expanding the numerator and the denominator and cancelling x or any power of x which may occur as a common factor, then make $x \to 0$ and get the required result.

Consider the limit of $y=\dfrac{(1+x)^3-1-3x}{x^2}$ when $x \to 0$.

$$\frac{(1+x)^3-1-3x}{x^2}=\frac{3x^2+x^3}{x^2}=3+x.$$

$$\therefore \quad \underset{x\to 0}{\mathrm{Lt}}\frac{(1+x)^3-1-3x}{x^2}=3.$$

3. If we require Lt $f(x)$ when $x \to \infty$, we put $x=\dfrac{1}{y}$ and proceed to find the limit as $y \to 0$.

Find $\underset{x\to\infty}{\text{Lt}} \dfrac{ax^2+bx+c}{a_1x^2+b_1x+c_1}$.

Putting $x=\dfrac{1}{y}$, we get

$$\underset{x\to\infty}{\text{Lt}} \frac{ax^2+bx+c}{a_1x^2+b_1x+c_1} = \underset{y\to 0}{\text{Lt}} \frac{\dfrac{a}{y^2}+\dfrac{b}{y}+c}{\dfrac{a_1}{y^2}+\dfrac{b_1}{y}+c_1}$$

$$= \underset{y\to 0}{\text{Lt}} \frac{a+by+cy^2}{a_1+b_1y+c_1y^2} = \frac{a}{a_1}.$$

§ 9. Some general theorems on limits.

The reader will have no difficulty in verifying the following results :—

If $f(x)\to\alpha$ and $\phi(x)\to\beta$ as $x\to a$, then

1. $\underset{x\to a}{\text{Lt}} \{f(x)\pm\phi(x)\} = \alpha\pm\beta$.

2. $\underset{x\to a}{\text{Lt}} \{f(x)\,\phi(x)\} = \alpha\beta$.

3. $\underset{x\to a}{\text{Lt}} \dfrac{f(x)}{\phi(x)} = \dfrac{\alpha}{\beta}$ provided $\beta\neq 0$.

4. $\underset{x\to a}{\text{Lt}} f\{\phi(x)\} = f\{\phi(a)\}$.

§ 10. Certain special limits.

(1) $\underset{x\to a}{\text{Lt}} \dfrac{x^n-a^n}{x-a} = n\,a^{n-1}$ for all rational values of n.

Case (*i*). Let n be a positive integer.

Substitute $a+h$ for x and we get

$$\frac{x^n-a^n}{x-a} = \frac{(a+h)^n-a^n}{(a+h)-a} = \frac{(a+h)^n-a^n}{h}.$$

Expanding the numerator by Binomial Theorem, we get

$$\frac{x^n-a^n}{x-a} = \frac{a^n+n\,a^{n-1}h+\dfrac{n(n-1)}{2\,!}a^{n-2}h^2+\ldots+h^n-a^n}{h}$$

$$= n\,a^{n-1}+\frac{n(n-1)}{2\,!}a^{n-2}h+\ldots+h^{n-1}.$$

All the other terms except the first term contain h. As $x\to a$, $h\to 0$.

$$\therefore \operatorname*{Lt}_{x \to a} \frac{x^n - a^n}{x - a} = \operatorname*{Lt}_{h \to 0} \left\{ n\, a^{n-1} + \frac{n(n-1)}{2!} a^{n-2} h + \dots + h^{n-1} \right\}$$

$= n \,.\, a^{n-1}$ (as there are only a finite number of terms).

Case (*ii*). Let $n = \frac{p}{q}$, where p and q are positive integers.

Let $x = y^q$ and $a = b^q$.

Then, as $x \to a$, $y^q \to b^q$.

So, as $x \to a$, $y \to b$.

$$\frac{x^n - a^n}{x - a} = \frac{x^{p/q} - a^{p/q}}{x - a} = \frac{(y^q)^{p/q} - (b^q)^{p/q}}{y^q - b^p} = \frac{y^p - b^p}{y^q - b^q}.$$

$$\therefore \operatorname*{Lt}_{x \to a} \frac{x^n - a^n}{x - a} = \operatorname*{Lt}_{y \to b} \frac{y^p - b^p}{y^q - b^q}$$

$$= \operatorname*{Lt}_{y \to b} \frac{y^p - b^p}{y - b} \div \frac{y^q - b^q}{y - b}$$

$$= \operatorname*{Lt}_{y \to b} \frac{y^p - b^p}{y - b} \div \operatorname*{Lt}_{y \to b} \frac{y^q - b^q}{y - b}$$

$= p \,.\, b^{p-1} \div q \,.\, b^{q-1}$ by case (*i*)

$$= \frac{p}{q} \,.\, b^{p-q}$$

$$= \frac{p}{q} \,.\, b^{q\left(\frac{p}{q}-1\right)}$$

$$= n \,.\, a^{n-1}.$$

Case (*iii*). $n = -m$ where m is a positive integer or fraction,

$$\frac{x^n - a^n}{x - a} = \frac{x^{-m} - a^{-m}}{x - a} = \frac{a^m - x^m}{x^m a^m (x - a)}$$

$$= -\frac{1}{x^m a^m} \cdot \frac{x^m - a^m}{x - a}.$$

$$\therefore \operatorname*{Lt}_{x \to c} \frac{x^n - a^n}{x - a} = \operatorname*{Lt}_{x \to a} \left(-\frac{1}{x^m a^m} \cdot \frac{x^m - a^m}{x - a} \right)$$

$$= -\frac{1}{a^m} \operatorname*{Lt}_{x \to a} \left(\frac{1}{x^m} \right) \cdot \operatorname*{Lt}_{x \to a} \frac{x^m - a^m}{x - a}$$

$= -\frac{1}{a^m} \cdot \frac{1}{a^m} \cdot m \,.\, a^{m-1}$ by cases (*i*) and (*ii*)

$$= -m \,.\, a^{-m-1} = n \,.\, a^{n-1}.$$

Hence $\underset{x\to a}{\text{Lt}} \dfrac{x^n - a^n}{x - a} = n \cdot a^{n-1}$ for all rational values of n.

Example.

Find $\underset{x\to a}{\text{Lt}} \dfrac{x^{5/8} - a^{5/8}}{x^{1/3} - a^{1/3}}$.

$$\underset{x\to a}{\text{Lt}} \frac{x^{5/8} - a^{5/8}}{x^{1/3} - a^{1/3}} = \underset{x\to a}{\text{Lt}} \frac{x^{5/8} - a^{5/8}}{x - a} \div \underset{x\to a}{\text{Lt}} \frac{x^{1/3} - a^{1/3}}{x - a}$$

$$= \tfrac{5}{8} \cdot a^{5/8-1} \div \tfrac{1}{3} \cdot a^{1/3-1}$$

$$= \frac{15}{8} a^{7/24}.$$

(2) $\underset{\theta\to 0}{\text{Lt}} \dfrac{\sin\theta}{\theta} = 1$ (θ is measured in radians).

Describe a circle with O as centre and r as radius. In this circle let AOP be θ radians. Draw the perpendicular PN from P to OA and the tangent at the point A meeting OP produced at T.

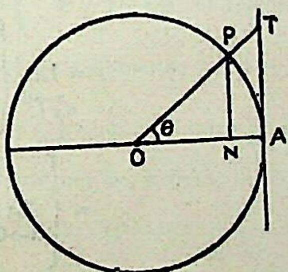

Fig. 1

In the figure, we see that

Area of $\triangle AOP <$ area of sector $AOP <$ area of $\triangle OAT$

i.e., $\frac{1}{2}\, OA \cdot PN < \frac{1}{2} \cdot OP^2 \cdot \theta < \frac{1}{2}\, OA \cdot AT$

i.e., $\frac{1}{2}\, r \cdot r\sin\theta < \frac{1}{2}\, r^2 \cdot \theta < \frac{1}{2}\, r \cdot r\tan\theta$.

Dividing throughout by $\frac{1}{2}\, r^2 \sin\theta$ (which is +ve), we get

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

$$\therefore\ 1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

$\therefore\ \dfrac{\sin\theta}{\theta}$ lies between 1 and $\cos\theta$.

$\therefore\ \underset{\theta\to 0}{\text{Lt}} \dfrac{\sin\theta}{\theta}$ lies between 1 and the limit of $\cos\theta$ as $\theta \to 0$.

As $\theta \to 0$, $\cos\theta \to 1$.

$$\therefore \quad \underset{\theta \to 0}{\text{Lt}} \frac{\sin\theta}{\theta} = 1.$$

Examples.

Ex. 1. Find $\underset{\theta \to 0}{\text{Lt}} \dfrac{\tan\theta}{\theta}$.

$$\underset{\theta \to 0}{\text{Lt}} \frac{\tan\theta}{\theta} = \underset{\theta \to 0}{\text{Lt}} \frac{\sin\theta}{\theta} \cdot \frac{1}{\cos\theta}$$

$$= \underset{\theta \to 0}{\text{Lt}} \frac{\sin\theta}{\theta} \cdot \underset{\theta \to 0}{\text{Lt}} \frac{1}{\cos\theta} = 1.$$

Ex. 2. Find $\underset{x \to 0}{\text{Lt}} \dfrac{1-\cos x}{x}$.

$$\underset{x \to 0}{\text{Lt}} \frac{1-\cos x}{x} = \underset{x \to 0}{\text{Lt}} \frac{2\sin^2\frac{x}{2}}{x}$$

$$= \underset{x \to 0}{\text{Lt}} \left\{\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right\}^2 \cdot \frac{x}{2}$$

$$= \left\{\underset{x \to 0}{\text{Lt}} \frac{\sin\frac{x}{2}}{\frac{x}{2}}\right\}^2 \underset{x \to 0}{\text{Lt}} \frac{x}{2}$$

$$= 1 \times 0 = 0.$$

Ex. 3. Find $\underset{x \to 0}{\text{Lt}} \dfrac{1-\cos x}{x^2}$.

$$\underset{x \to 0}{\text{Lt}} \frac{1-\cos x}{x^2} = \underset{x \to 0}{\text{Lt}} \frac{2\sin^2\frac{x}{2}}{x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \left\{\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right\}^2 \times \tfrac{1}{2} = 1 \times \tfrac{1}{2} = \tfrac{1}{2}.$$

(3) $\underset{n \to \infty}{\text{Lt}} \left(1 + \dfrac{1}{n}\right)^n$.

Case (*i*). Let n be a positive integer.

Expanding $\left(1+\frac{1}{n}\right)^n$ by Binomial Theorem, we get

$$\left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \ldots\ldots$$
$$\ldots\ldots + \frac{n(n-1)\ldots\ldots(n-n+1)}{n!}\cdot\frac{1}{n^n}$$
$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) +$$
$$\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots\ldots$$
$$\ldots\ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\frac{n-1}{n}\right).$$

There are $(n+1)$ terms in the expansion. As the value of n increases, the number of terms in the expansion and the value of each term after the second is increasing. Therefore as n increases, the value of $\left(1+\frac{1}{n}\right)^n$ also increases.

Since each term of the expansion in $\left(1+\frac{1}{n}\right)^n$ is positive and since the value of $\left(1+\frac{1}{n}\right)^n$ increases with n, $\left(1+\frac{1}{n}\right)^n$ tends to infinity or to a finite number as n tends to infinity. Each term after the second term in the expansion of $\left(1+\frac{1}{n}\right)^n$ is less than the corresponding term in the series

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots + \frac{1}{n!}$$

$$\therefore \left(1+\frac{1}{n}\right)^n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots + \frac{1}{n!}$$
$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots + \frac{1}{2^{n-1}}$$
$$i.e., < 1 + 2\left(1-\frac{1}{2^n}\right)$$
$$< 3.$$

$\therefore$ The limit of $\left(1+\frac{1}{n}\right)^n$ cannot be infinity.

$$\therefore \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^n = \text{a finite number.}$$

This finite number is usually denoted by the letter e.

Case (*ii*). Let n be a positive fraction.

$\therefore$ n lies between two consecutive integers.

Let those numbers be m and $m+1$.

$$m < n < m+1.$$

$$\therefore \quad \frac{1}{m+1} < \frac{1}{n} < \frac{1}{m}$$

i.e., $$1+\frac{1}{m+1} < 1+\frac{1}{n} < 1+\frac{1}{m}.$$

Since $m < n < m+1$,

$$\left(1+\frac{1}{m+1}\right)^m < \left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{m}\right)^{m+1}.$$

As n tends to infinity, m and $m+1$ also tend to infinity.

$$\underset{m\to\infty}{\text{Lt}} \left(1+\frac{1}{m}\right)^{m+1} = \underset{m\to\infty}{\text{Lt}} \left(1+\frac{1}{m}\right)^m \left(1+\frac{1}{m}\right)$$

$$= \underset{m\to\infty}{\text{Lt}} \left(1+\frac{1}{m}\right)^m \underset{m\to\infty}{\text{Lt}} \left(1+\frac{1}{m}\right)$$

$$= e \,.\, 1 \quad [\because \ m \text{ is a +ve integer}]$$

$$= e.$$

$$\underset{m\to\infty}{\text{Lt}} \left(1+\frac{1}{m+1}\right)^m = \underset{m\to\infty}{\text{Lt}} \left(1+\frac{1}{m+1}\right)^{m+1} \div \underset{m\to\infty}{\text{Lt}} \left(1+\frac{1}{m+1}\right)$$

$$= e.$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^n = e.$$

Case (*iii*). Let n be a negative index.

Let $n = -m$ where m is +ve.

$$\left(1+\frac{1}{n}\right)^n = \left(1-\frac{1}{m}\right)^{-m}$$

$$= \left(\frac{m-1}{m}\right)^{-m}$$

$$= \left(\frac{m}{m-1}\right)^m$$

$$= \left(1+\frac{1}{m-1}\right)^{m-1} \left(1+\frac{1}{m-1}\right).$$

§ 12. Theorems on continuous functions.

The following theorems are of constant application :—

THEOREM 1. If $f(x)$ is continuous at a and if $f(a)$ is not zero, then for values of x near a, $f(x)$ has the same sign as $f(a)$.

THEOREM 2. If $f(x)$ be continuous in the interval (a, b) and if $f(a) = A$ and $f(b) = B$, then $f(x)$ will assume once at least every value lying between A and B as x varies continuously from a to b ; in particular if A and B have opposite signs, $f(x)$ will become zero for at least one value of x between a and b.

(The proofs of the above theorems are given in any book of Algebra.)

CHAPTER II

DIFFERENTIATION

§ 13. Definition.

Let y be a continuous function of x; then an increase in the value of x will produce an increase or a decrease in the value of y. These increments are generally denoted by the symbols Δx, Δy respectively; Δy is positive or negative according as y increases or decreases and similarly for Δx. If Δx, the increment in x is indefinitely small, Δy the corresponding increment in y, will also be indefinitely small since y is a continuous function of x; usually the average rate of change of y with respect to x, *i.e.*, the ratio $\frac{\Delta y}{\Delta x}$ tends to a definite limit as Δx tends to zero. This limit, when it exists, is called the differential coefficient of y with respect to x and is denoted by the symbol $\frac{dy}{dx}$.

So, if $y = f(x)$, then $y + \Delta y = f(x + \Delta x)$

$$\frac{dy}{dx} = \underset{x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x}$$

$$= \underset{\Delta x \to 0}{\text{Lt}} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

The differential coefficient is also called the derivative and is sometimes denoted by the symbol $D_x y$ or to read as Dy when there is no doubt about the independent variable. The differential coefficient of $f(x)$ is written generally as $\frac{d}{dx} f(x)$ or $D f(x)$ or $f'(x)$.

STANDARD FORMS

§ 14·1. Differential coefficient of x^n.

Let y be equal to x^n. When x receives an increment Δx, let Δy denote the corresponding increment in y.

Then $y + \Delta y = (x + \Delta x)^n$.

$$\therefore \quad \Delta y = (x + \Delta x)^n - x^n.$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^n - x^n}{\Delta x}.$$

$$\therefore \frac{dy}{dx} = \underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x}$$

$$= \underset{\Delta x \to 0}{\text{Lt}} \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

$$= \underset{x + \Delta x \to x}{\text{Lt}} \frac{(x + \Delta x)^n - x^n}{(x + \Delta x) - x}$$

$= n \,.\, x^{n-1}$ for values of n by § 10 (1).

$$\therefore \frac{d}{dx}(x^n) = nx^{n-1}.$$

§ 14·2. Differential coefficient of e^x.

Let y be e^x. Corresponding to an increment Δx in x, let the increment in y be Δy.

Then $y + \Delta y = e^{x+\Delta x}$.

$$\therefore \Delta y = e^{x+\Delta x} - e^x.$$

$$\frac{dy}{dx} = \underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x}$$

$$= \underset{\Delta x \to 0}{\text{Lt}} \frac{e^{x+\Delta x} - e^x}{\Delta x}$$

$$= \underset{\Delta x \to 0}{\text{Lt}} \frac{e^x (e^{\Delta x} - 1)}{\Delta x}$$

$$= e^x \,.\, \underset{\Delta x \to 0}{\text{Lt}} \frac{e^{\Delta x} - 1}{\Delta x}$$

$= e^x$ by Ex. 2, page 15.

$$\therefore \frac{d}{dx}(e^x) = e^x.$$

§ 14·3. Differential coefficient of $\log_e x$.

Let y be $\log_e x$. Let the increment in y corresponding to an increment Δx in x be Δy.

Then $y + \Delta y = \log_e (x + \Delta x)$.

$$\frac{dy}{dx} = \underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x}$$

$$= \underset{\Delta x \to 0}{\text{Lt}} \frac{\log_e (x + \Delta x) - \log_e x}{\Delta x}$$

$$= \underset{\Delta x \to 0}{\text{Lt}} \frac{\log_e \left(1 + \frac{\Delta x}{x}\right)}{\Delta x}.$$

Substitute h for $\dfrac{\Delta x}{x}$.

As $\Delta x \to 0$, $h \to 0$.

$$\therefore \quad \frac{dy}{dx} = \underset{h\to 0}{\text{Lt}} \frac{\log_e (1+h)}{xh}$$

$$= \frac{1}{x} \underset{h\to 0}{\text{Lt}} \log_e (1+h)^{1/h}$$

$$= \frac{1}{x} \log_e e = \frac{1}{x}.$$

§ 14·4. Differential coefficient of sin x.

If $y = \sin x$,

$$y + \Delta y = \sin (x + \Delta x).$$

$$\therefore \quad \frac{dy}{dx} = \underset{\Delta x\to 0}{\text{Lt}} \frac{\sin (x+\Delta x) - \sin x}{\Delta x}$$

$$= \underset{\Delta x\to 0}{\text{Lt}} \frac{2 \sin \frac{\Delta x}{2} . \cos \left(x + \frac{\Delta x}{2}\right)}{\Delta x}$$

$$= \underset{\Delta x\to 0}{\text{Lt}} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} . \cos \left(x + \frac{\Delta x}{2}\right)$$

$$= \underset{\Delta x\to 0}{\text{Lt}} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} . \lim_{\Delta x\to 0} \cos \left(x + \frac{\Delta x}{2}\right)$$

$= 1 . \cos x$ (as x is measured in radians).

$$\therefore \quad \frac{d}{dx}(\sin x) = \cos x.$$

§ 14·5. Differential coefficient of cos x.

If $y = \cos x$, then $y + \Delta y = \cos (x + \Delta x)$.

$$\frac{dy}{dx} = \underset{\Delta x\to 0}{\text{Lt}} \frac{\cos (x+\Delta x) - \cos x}{\Delta x}$$

$$= - \underset{\Delta x\to 0}{\text{Lt}} \frac{2 \sin \frac{\Delta x}{2} . \sin \left(x + \frac{\Delta x}{2}\right)}{\Delta x}$$

$$= -\underset{\Delta x \to 0}{\mathrm{Lt}} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \underset{\Delta x \to 0}{\mathrm{Lt}} \sin\left(x + \frac{\Delta x}{2}\right)$$

$= -1 \times \sin x$, (x being in circular measure).

$$\therefore \frac{d}{dx}(\cos x) = -\sin x.$$

§ 14·6. Differential coefficient of tan x.

If $y = \tan x$, then $y + \Delta y = \tan(x + \Delta x)$.

$$\frac{dy}{dx} = \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{\tan(x + \Delta x) - \tan x}{\Delta x}$$

$$= \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{\sin(x + \Delta x)\cos x - \sin x \cos(x + \Delta x)}{\cos(x + \Delta x)\cos x \,.\, \Delta x}$$

$$= \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{\sin \Delta x}{\Delta x} \cdot \frac{1}{\cos(x + \Delta x)} \cdot \frac{1}{\cos x}$$

$$= \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{\sin \Delta x}{\Delta x} \,.\, \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{1}{\cos(x + \Delta x)} \cdot \frac{1}{\cos x}$$

$$= 1 \,.\, \frac{1}{\cos x} \cdot \frac{1}{\cos x}$$

$= \sec^2 x$, (as x is measured in radians).

$$\therefore \frac{d}{dx}(\tan x) = \sec^2 x.$$

§ 15·1. General theorems on differential coefficients.

The differential coefficient of a constant is zero.

If $y = c$, then $y + \Delta y = c$, since an increase in the value of x produces no change in the value of a constant.

$\therefore \quad \Delta y = 0.$

$$\frac{dy}{dx} = \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{0}{\Delta x} = 0.$$

§ 15·2. Differential coefficient of the product of a constant and a function.

Let $y = cu$, where c is a constant and u a function of x.

Let Δy and Δu be the increments in y and u respectively corresponding to an increment Δx in x.

Then $y + \Delta y = c(u + \Delta u)$.

$$\therefore \quad \Delta y = c \, \Delta u.$$

$$\frac{dy}{dx} = \underset{\Delta x \to 0}{\text{Lt}}\ c \cdot \frac{\Delta u}{\Delta x} = c \underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta u}{\Delta x} = c\frac{du}{dx}.$$

Hence the differential coefficient of the product of a constant and a function is equal to the product of the constant and the differential coefficient of the function.

§ 15·3. Differential coefficient of a sum (or difference).

Let $y = u + v - w$, where u, v, w are functions of x.

Let Δy, Δu, Δv, Δw be the increments in y, u, v, w respectively corresponding to an increment Δx in x.

Then $y + \Delta y = u + \Delta u + v + \Delta v - (w + \Delta w)$.

$\therefore \quad \Delta y = \Delta u + \Delta v - \Delta w.$

$\therefore \quad \dfrac{\Delta y}{\Delta x} = \dfrac{\Delta u}{\Delta x} + \dfrac{\Delta v}{\Delta x} - \dfrac{\Delta w}{\Delta x}.$

When $\Delta x \to 0$, we have

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}.$$

Hence the differential coefficient of the sum of a finite number of functions is equal to the sum of the differential coefficients of those functions.

For example,

the differential coefficient of $ax^2 + 2bx + c$ is $2ax + 2b$.

$$\text{Similarly, } \frac{d}{dx}(\sin x + k\cos x) = \frac{d}{dx}(\sin x) + \frac{d}{dx}(k\cos x)$$

$$= \cos x + k\frac{d}{dx}(\cos x)$$

$$= \cos x - k\sin x.$$

$$\frac{d}{dx}(a - bx)^2 = \frac{d}{dx}(a^2 - 2abx + b^2x^2) = -2ab + 2b^2x.$$

$$\frac{d}{dx}(x^{1/2} - 2)^2 = \frac{d}{dx}(x - 4x^{1/2} + 4) = 1 - 4 \times \tfrac{1}{2}x^{-1/2} = 1 - \frac{2}{x^{1/2}}.$$

Exercises III.

Differentiate the following functions with respect to x :—

1. $x^2 - 3x + 2$.
2. $4x^2 - 9x - 3$.
3. $lx^3 + mx^2 + nx + k$.
4. $4x^3 - 9 + 6x^2$.
5. $6x^9 - 2x + \dfrac{1}{x}$.
6. $ax^{2n} + bx^n + c$.
7. $\dfrac{3}{x^3} - \dfrac{2}{x^2} + \dfrac{1}{x} + 4$.

8. $\dfrac{x^3+4x^2+3}{x^2}$.

9. $\sqrt[3]{x^2}$.

10. $\dfrac{x^3+x+1}{\sqrt{x}}$.

11. $\dfrac{5}{x^{2/3}}+\dfrac{3}{x^{1/4}}-\sqrt{x}$.

12. $(x^2-3)(2x+1)$.

13. $\left(\dfrac{2x-3}{x^2}\right)^3$.

14. $\left(\dfrac{1}{x}-x\right)^3$.

15. $\left(x^{1/2}-2x\right)\left(x-\dfrac{1}{x}\right)$.

16. $e^x+\sin x$.

17. $5\sin x+\log x$.

18. $3\log x-e^x-7\cos x$.

19. $5e^x-\log x+\sqrt[3]{x^2}$.

20. $\frac{1}{2}e^x-x^m+2$.

21. $\log\left(\dfrac{x}{e^x}\right)$.

22. $4\log x^2+\sin x$.

23. $\sqrt{2}\sin x+4x^5-\dfrac{3}{x^4}$.

24. $(ax)^n+b^n\log x$.

25. $4x^2-3\cos x+e^x+2\sin x$.

§ 15·4. Product Rule.

Let $y=uv$ where u and v are functions of x. Let Δy, Δu, Δv be the increments in y, u, v respectively corresponding to an increment Δx in x.

Then $y+\Delta y=(u+\Delta u)(v+\Delta v)$.

$$\therefore\quad \Delta y=(u+\Delta u)(v+\Delta v)-uv$$
$$=u\,\Delta v+v\,\Delta u+\Delta u\,\Delta v.$$
$$\therefore\quad \frac{\Delta y}{\Delta x}=u\frac{\Delta v}{\Delta x}+v\frac{\Delta u}{\Delta x}+\frac{\Delta u}{\Delta x}\Delta v.$$

When Δx and therefore Δu, Δv, Δy all tend to zero, $\dfrac{\Delta y}{\Delta x}, \dfrac{\Delta u}{\Delta x}, \dfrac{\Delta v}{\Delta x}$ tend to $\dfrac{dy}{dx}, \dfrac{du}{dx}, \dfrac{dv}{dx}$ respectively.

$\dfrac{\Delta u}{\Delta x}\Delta v$ will tend to $\dfrac{du}{dx}\times 0$, *i.e.*, to zero.

$$\therefore\quad \mathbf{\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}}.$$

Hence the differential coefficient of the product of two functions = first function × differential coefficient of the second function + the second function × differential coefficient of the first function.

Examples.

Ex. 1. $$\frac{d}{dx}\left\{(x^2+1)(x+2)\right\}=(x^2+1)\frac{d}{dx}(x+2)+(x+2)\frac{d}{dx}(x^2+1)$$
$$=(x^2+1)+(x+2)\,2x$$
$$=3x^2+4x+1.$$

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Ex. 2. $\frac{d}{dx}\left\{\sqrt{x}\,(x^2+2)\right\}$

$$= \sqrt{x}\frac{d}{dx}(x^2+2) + (x^2+2)\frac{d}{dx}(\sqrt{x})$$
$$= \sqrt{x}\,.\,2x + (x^2+2)\,.\,\tfrac{1}{2}\,.\,x^{-1/2}$$
$$= 2x^{3/2} + \frac{x^2+2}{2x^{1/2}} = \frac{5x^2+2}{2x^{1/2}}.$$

Ex. 3. $\frac{d}{dx}(e^x \sin x) = e^x\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(e^x)$

$$= e^x \cos x + \sin x\,.\,e^x = e^x(\cos x + \sin x).$$

Ex. 4. $\frac{d}{dx}\left\{3x^5 \log x\right\} = 3\,.\left\{x^5\frac{d}{dx}(\log x) + \log x\,.\,\frac{d}{dx}(x^5)\right\}$

$$= 3x^5\,.\,\frac{1}{x} + 3\log x\,.\,5x^4 = 3x^4 + 15x^4 \log x.$$

§ 15·5. If $y = uvw$, where u, v, w are functions of x, we get $\frac{dy}{dx} = uv\frac{dw}{dx} + w\frac{d}{dx}(uv)$ considering uv as a single function

$$= uv\frac{dw}{dx} + w\left(u\frac{dv}{dx} + v\frac{du}{dx}\right)$$
$$= uv\frac{dw}{dx} + wu\frac{dv}{dx} + wv\frac{du}{dx}.$$

(*Note*: To get the derivative of the product of 3 functions, multiply the derivative of each by the rest and add.)

Dividing by uvw, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}.$$

Similarly, if $y = u_1, u_2, v_3 \ldots u_n$, where $u_1, u_2, u_3 \ldots u_n$ are functions of x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u_1}\frac{du_1}{dx} + \frac{1}{u_2}\frac{du_2}{dx} \ldots + \frac{1}{u_n}\frac{du_n}{dx}.$$

Examples.

Ex. 1. $\frac{d}{dx}\left\{x(x^2-1)(x^2+4)\right\}$.

Let $y = x(x^2-1)(x^2+4)$.

By § 15·5. $\frac{dy}{dx} = (x^2-1)(x^2+4) + 2x^2(x^2+4) + 2x^2(x^2-1)$.

Ex. 2. $\dfrac{d}{dx}(e^x \sin x \log x)$.

Let $y = e^x \sin x \log x$.

By § 15·5. $\dfrac{dy}{dx} = e^x \sin x \log x + e^x \cos x \log x + \dfrac{e^x \sin x}{x}$.

§ 15·6. Quotient Rule.

Let $y = \dfrac{u}{v}$, where u and v are functions of x. When x becomes $x + \Delta x$, let y, u, v become $y + \Delta y$, $u + \Delta u$, $v + \Delta v$ respectively.

$$\therefore \quad y + \Delta y = \frac{u + \Delta u}{v + \Delta v}.$$

$$\therefore \quad \Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}$$

$$= \frac{v\,\Delta u - u\,\Delta v}{v\,(v + \Delta v)}.$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v^2 + v\,\Delta v}.$$

In the limit, when $\Delta x \to 0$ and therefore Δu, Δv and Δy tend to zero, we get

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Hence the derivative of a quotient

$$= \frac{\text{denominator} \times \text{derivative of numerator} - \text{numerator} \times \text{derivative of denominator}}{(\text{denominator})^2}$$

Examples.

Ex. 1. $\dfrac{d}{dx}\left(\dfrac{x^3}{3x-2}\right)$

$$= \frac{(3x-2)\dfrac{d}{dx}(x^3) - x^3\dfrac{d}{dx}(3x-2)}{(3x-2)^2}$$

$$= \frac{(3x-2)\,3x^2 - x^3 \times 3}{(3x-2)^2}$$

$$= \frac{6x^2(x-1)}{(3x-2)^2}.$$

Ex. 2. $\dfrac{d}{dx}\left(\dfrac{\sqrt{x}}{2x+3}\right)$

$$= \frac{(2x+3)\frac{d}{dx}(\sqrt{x}) - \sqrt{x}\,\frac{d}{dx}(2x+3)}{(2x+3)^2}$$

$$= \frac{(2x+3)\,\frac{1}{2}\,x^{-1/2} - x^{1/2}\,.\,2}{(2x+3)^2}$$

$$= \frac{(2x+3) - 4x}{2x^{1/2}\,(2x+3)^2}$$

$$= \frac{3-2x}{2\sqrt{x}\,(2x+3)^2}.$$

Ex. 3. $\dfrac{d}{dx}\left(\dfrac{\log x}{\sin x}\right)$

$$= \frac{\sin x\,\frac{d}{dx}(\log x) - \log x\,\frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{\frac{\sin x}{x} - \log x\,.\,\cos x}{\sin^2 x}$$

$$= \frac{\sin x - x\log x\,.\,\cos x}{x\sin^2 x}.$$

§ 15·7. **Differential coefficients of tan x, cot x, sec x, cosec x.**

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x\,\frac{d}{dx}(\sin x) - \sin x\,\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x\cos x - \sin x\,(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \sec^2 x.$$

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$

$$= \frac{\sin x\,\frac{d}{dx}(\cos x) - \cos x\,\frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{\sin x \times (-\sin x) - \cos x \times \cos x}{\sin^2 x}$$

$$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x}$$

$$= -\operatorname{cosec}^2 x.$$

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$= \frac{\cos x \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \times 0 - 1(-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \tan x \sec x.$$

$$\frac{d}{dx}(\operatorname{cosec} x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

$$= \frac{\sin x \frac{d}{dx}(1) - 1 \frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{\sin x \times 0 - 1 \times \cos x}{\sin^2 x}$$

$$= -\frac{\cos x}{\sin^2 x}$$

$$= -\cot x \operatorname{cosec} x.$$

(*Note* : The student is advised to obtain the last three derivatives from first principles.)

Exercises IV.

Differentiate the following functions :—

1. $(x-1)(3x-1)$.
2. $(1+x^2)(1-2x^2)$.
3. $x^2 \sin x$.
4. $(7x-3)(3x^2+4)$.
5. $\cos x \log x$.
6. $2x \cos x - x^2 \sin x$.
7. $8\sqrt{x} \log x$.
8. $(2x-1)(3x+7)(4x^2-3)$.
9. $x e^x \sin x$.
10. $x(x+3)(x^2+9)$.
11. $\frac{\sin x}{x}$.
12. $\frac{\cos x}{x^2}$.
13. $\frac{1+x^2}{1-x^2}$.
14. $\frac{3 \operatorname{cosec} x + 2}{7 + 3 \cot x}$.
15. $\frac{\sin x}{1+\tan x}$.

16. $\dfrac{3+2\tan x}{4x-9}$.

17. $\dfrac{(x+3)(x-2)}{(x-3)(x+2)}$.

18. $\dfrac{(x-1)(x-2)}{(x+1)(x+2)}$.

19. $\dfrac{x^4-6+x^{-3}}{2x-1+x^{-1}}$.

20. $\dfrac{e^x+\sin x}{\sec x-x^5}$.

21. $\dfrac{\log x+2e^x}{1+\sec x}$.

§ 15·8. Function of function rule.

We shall explain first what is meant by a function of a function. When y is a function of u, say, $y=f(u)$ and u is a function of x, say, $u=\phi(x)$, then y is said to be a function of a function. Functions of functions are of constant occurrence in calculus.

For example,

(1) consider the function $y=u^{1/3}$ where $u=x^2-4x+8$.

$\therefore$ y is a function of a function of x indirectly through u.

(2) consider $y=\sin u$ where $u=3x^2+4$.

$\therefore$ y is a function of a function of x.

(3) consider $y=\sin^3(4x-3)$.

Put $4x-3=u$, $\sin u=v$.

$\therefore$ $y=v^3$ where $v=\sin u$ and $u=4x-3$.

Similar examples of functions of a function are $\log(\sin x)$, $\sin(e^x)$ ...

Consider the function $y=\sin(x^2)$. The differential coefficient of the function cannot be conveniently worked out at once from the first principles but may be obtained by two stages by denoting x^2 by u and by writing $y=\sin u$.

Generally let y be a function of u where u is a function of x; *i.e.*, $y=f(u)$ where $u=\phi(x)$.

When x takes the increment Δx, let the corresponding increment in u be Δu; when u takes the increment Δu, let y take the consequent increment Δy.

Hence when x takes the increment Δx, y takes the increment Δy and when Δx tends to zero, Δy and Δu also tend to zero simultaneously.

Now $\dfrac{\Delta y}{\Delta x}=\dfrac{\Delta y}{\Delta u}\cdot\dfrac{\Delta u}{\Delta x}$.

$$\therefore \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{\mathrm{Lt}} \left(\frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \right)$$

$$= \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{\Delta y}{\Delta u} \cdot \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{\Delta u}{\Delta x}$$

$$= \underset{\Delta u \to 0}{\mathrm{Lt}} \frac{\Delta y}{\Delta u} \cdot \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{\Delta u}{\Delta x}, \text{ as when } \left.\begin{matrix} \Delta x \to 0, \\ \Delta u \to 0 \end{matrix}\right\}$$

$$= \frac{dy}{du} \cdot \frac{du}{dx}.$$

For instance, in the example mentioned above,

$y = \sin u$ and $u = x^2$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= \cos u \,.\, 2x$$
$$= \cos (x^2) \,.\, 2x.$$

This result may be extended to any number of functions.

If $y = f(u)$, $u = F(v)$ and $v = \phi(x)$,

then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dv} \cdot \dfrac{dv}{dx}.$

Examples.

Ex. 1. $y = (2x^2 + 4)^3$.

Put $2x^2 + 4 = u$.

Then $y = u^3$ where $u = 2x^2 + 4$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \,.\, 4x$$
$$= 3\,(2x^2 + 4)^2 \,.\, 4x = 48x\,(x^2 + 2)^2.$$

Ex. 2. $y = \sin (ax + b)$.

Put $ax + b = u$.

Then $y = \sin u$ where $u = ax + b$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \,.\, a = a \cos (ax + b).$$

Ex. 3. $y = \cos^3 x$.

Put $\cos x = u$.

Then $y = u^3$ where $u = \cos x$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= 3u^2 (-\sin x)$$
$$= -3 \cos^2 x \sin x.$$

Ex. 4. $y = \dfrac{1}{\sqrt{3+2x}}$.

Put $3+2x = u$.

Then $y = \dfrac{1}{\sqrt{u}} = u^{-1/2}$ where $u = 3+2x$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\tfrac{1}{2}\, u^{-3/2}\, 2$$

$$= -\frac{1}{(3+2x)^{3/2}}.$$

Ex. 5. $y = \sin^2 2x$.

Put $2x = u$ and $\sin u = v$.

$\therefore\ y = v^2$ where $v = \sin u$ and $u = 2x$.

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = 2v \cdot \cos u \cdot 2$$

$$= 2 \sin 2x \cdot \cos 2x \cdot 2$$

$$= 4 \sin 2x \cdot \cos 2x = 2 \sin 4x.$$

Ex. 6. $y = \log(\tan e^x)$.

Put $e^x = u$ and $\tan u = v$.

$\therefore\ y = \log v$.

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{du}\frac{du}{dx} = \frac{1}{v}\sec^2 u\ e^x$$

$$= \frac{1}{\sin(e^x)\cos(e^x)}\, e^x.$$

Exercises V.

Differentiate with respect to x the functions :

1. $\sin 3x$.
2. $\tan^2 x$.
3. $\operatorname{cosec} \dfrac{x}{3}$.
4. $\sec^2 x$.
5. $\operatorname{cosec}^3 x$.
6. e^{2x}.
7. $\log(2x+3)$.
8. $\cos(4x-3)$.
9. $\cot^n x$.
10. $\sqrt{\tan x}$.
11. $\sin(b + ax^n)$.
12. $(ax^2 + bx + c)^{1/2}$.
13. $\sqrt[3]{(a+bx)^2}$.
14. $\cos(\log x)$.
15. $\sqrt{1 + \sin x}$.
16. $\tan(e^x)$.
17. $\sin(\cos x)$.
18. $\log(\cos x)$.
19. $(ax^2 + 2bx + c)^n$.
20. $(x+a)^m (x+b)^n$.
21. $(2x+1)^3 (x^2-1)^4$.

22. $(1-x)\sqrt{1+x}$.
23. $x(a-x)^n$.
24. $\dfrac{x}{\sqrt{2x-1}}$.
25. $\left(\dfrac{x}{x^2+1}\right)^n$.
26. $\sqrt{\dfrac{1-x}{1+x}}$.
27. $\sqrt{\dfrac{x^2+1}{x^2-1}}$.
28. $\sin^2 3x$.
29. $(3x+5)^4\sqrt{x^2-1}$.
30. $\cos^2 3x \cdot \sec 5x$.
31. $\dfrac{(a-x)^{n/2}}{(b+x)^n}$.
32. $\dfrac{1-\cos 2x}{1+\cos 2x}$.
33. $\sin^m x \cos^n x$.
34. $\sin mx \cos nx$.
35. $\cot^3 7x \cdot \cos x$.
36. $\operatorname{cosec} 2x \cdot \sec^2 4x$.
37. $\log\dfrac{1+\sin x}{1-\sin x}$.
38. $\log\dfrac{1+\cos x}{1-\sin x}$.
39. $e^{2x-3} \cdot \sin 2x$.
40. $\log\dfrac{b+e^{3x}}{a-e^x}$.
41. $e^{-x}(\sin x+2\cos x)$.
42. $\log(2x+3) \cdot e^{2x}\sin(4x)$.
43. $\sin(e^x \log x)$.
44. $\cos(\log x \cdot \cot x)$.
45. $\log(e^{-3x}\sin 4x)$.
46. $\cot(e^{4x}\operatorname{cosec} x)$.
47. $\log(\log x)$.
48. $x^2\log(\operatorname{cosec} x) \cdot e^{2x}$.
49. $e^{2x}\sin 3x\cos 4x$.
50. $x^3 e^{3x}\sin 8x$.

§ 15·9. Inverse functions.

If y is a continuous function of x, then x is generally a continuous function of y.

Let Δx and Δy be the corresponding increments of x and y, then evidently,

$$\frac{\Delta x}{\Delta y}\cdot\frac{\Delta y}{\Delta x}=1$$

and therefore when Δy and Δx tend to zero,

$$\frac{dx}{dy}\cdot\frac{dy}{dx}=1.$$

This result may be deduced from the differential coefficient of a function of a function.

If $y=f(u)$ when $u=\phi(x)$, we get

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

Put $y=x$. $\therefore$ $\dfrac{dx}{dx}=\dfrac{dx}{du}\cdot\dfrac{du}{dx}$.

$$\therefore \quad \frac{dx}{du} \cdot \frac{du}{dx} = 1.$$

$$\therefore \quad \frac{dx}{du} = \frac{1}{\frac{du}{dx}}.$$

Definition. If $\sin x = y$, then that angle x lying between $\pm \pi/2$ such that $\sin x = y$ is called the principal value of $\sin^{-1} y$.

Similarly the principal value of $\cos^{-1} y$ lies between 0 and π and that of $\tan^{-1} y$ lies between $\pm \pi/2$.

§ **15·10.** (*i*) *Differential coefficient of* $\sin^{-1} x$.

Let $\sin^{-1} x = y$; then $x = \sin y$.

Differentiating both sides with respect to y,

$$\frac{dx}{dy} = \cos y = + \sqrt{1 - x^2}$$ (taking into consideration the principal value of $\sin^{-1} x$).

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} \text{ since } \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

(*ii*) *Differential coefficient of* $\cos^{-1} x$.

Let $\cos^{-1} x = y$; then $x = \cos y$.

Differentiating with respect to y, we get

$$\frac{dx}{dy} = -\sin y = -\sqrt{1 - x^2}$$ (taking into account the principal value of $\cos^{-1} x$).

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = -\frac{1}{\sqrt{1 - x^2}}.$$

(*iii*) *Differential coefficient of* $\tan^{-1} x$.

Let $\tan^{-1} x = y$; then $x = \tan y$.

$$\frac{dx}{dy} = \sec^2 y.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + x^2}.$$

(*iv*) *Differential coefficient of* $\sec^{-1} x$.

Let $\sec^{-1} x = y$; then $x = \sec y$.

Differentiating both sides with respect to y, we get

$$\frac{dx}{dy} = \sec y \tan y.$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sec y \cdot \tan y} = \frac{1}{x\sqrt{x^2-1}}.$$

We can similarly prove that

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}.$$

Examples.

Ex. 1. Differentiate $\sin^{-1}(\sqrt{x})$.

Let $y = \sin^{-1}(\sqrt{x})$.

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{\sqrt{1-x}} \frac{1}{2} x^{-1/2}$$

$$= \frac{1}{2\sqrt{x(1-x)}}.$$

Ex. 2. Differentiate $(1+x^2)\tan^{-1} x$.

Let $y = (1+x^2)\tan^{-1} x$.

$$\therefore \frac{dy}{dx} = (1+x^2)\frac{d}{dx}(\tan^{-1} x) + \tan^{-1} x \cdot \frac{d}{dx}(1+x^2)$$

$$= (1+x^2)\frac{1}{1+x^2} + \tan^{-1} x \cdot 2x$$

$$= 1 + 2x\tan^{-1} x.$$

Ex. 3. Differentiate $\log \sec^{-1}(x^4)$.

Let $y = \log \sec^{-1}(x^4)$.

$$\therefore \frac{dy}{dx} = \frac{1}{\sec^{-1}(x^4)} \frac{d}{dx} \sec^{-1}(x^4)$$

$$= \frac{1}{\sec^{-1}(x^4)} \cdot \frac{1}{x^4\sqrt{x^8-1}} \cdot 4x^3$$

$$= \frac{4}{x\sqrt{x^8-1}\,\sec^{-1}(x^4)}.$$

Ex. 4. Differentiate $\tan^{-1}\dfrac{\cos x}{1+\sin x}$. (B.Sc. 52 M)

$$y = \tan^{-1}\frac{\cos x}{1+\sin x} = \tan^{-1}\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}$$

$$= \tan^{-1} \frac{2 \sin\left(\frac{\pi}{4} - \frac{x}{2}\right) \cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2 \cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}$$

$$= \tan^{-1} \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) = \frac{\pi}{4} - \frac{x}{2}.$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2}.$$

§ 15·11. Hyperbolic functions.

The expressions $\frac{1}{2}(e^x + e^{-x})$ and $\frac{1}{2}(e^x - e^{-x})$ are spoken of as hyperbolic cosine and sine respectively of the "argument" x and symbolically,

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x})$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

From these definitions we can easily find the following relations :—

(1) $\cosh^2 x - \sinh^2 x = \frac{1}{4}\{(e^x + e^{-x})^2 - (e^{-x} - e^{-x})^2\}.$

$$= \tfrac{1}{4}\{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}\} = 1.$$

(2) $2 \sinh x \cosh x = 2 . \dfrac{e^x - e^{-x}}{2} . \dfrac{e^x + e^{-x}}{2}$

$$= \tfrac{1}{2} . (e^{2x} - e^{-2x})$$

$$= \sinh 2x .$$

(3) $\cosh^2 x + \sinh^2 x = \frac{1}{4}\{(e^x + e^{-x})^2 + (e^x - e^{-x})^2\}$

$$= \tfrac{1}{4}\{(e^{2x} + 2 + e^{-2x} + e^{2x} - 2 + e^{-2x}\}$$

$$= \tfrac{1}{2} . (e^2x + e^{-2x})$$

$$= \cosh 2x.$$

§ 15·12. Inverse hyperbolic functions.

1. If $y = \sinh^{-1} x$, then $x = \sinh y$.

Then $\frac{1}{2}(e^y - e^{-y}) = x$

i.e., $e^{2y} - 2x e^y - 1 = 0$

i.e., $e^{2y} - 2x e^y + x^2 = x^2 + 1$

$\therefore \quad (e^y - x)^2 = x^2 + 1$

i.e., $e^y - x = \pm \sqrt{(x^2 + 1)}$.

Since e^y is always +ve, we can discard the negative sign.

$\therefore \quad e^y = x + \sqrt{x^2 + 1}.$

$\therefore \quad y = \log_e (x + \sqrt{x^2 + 1}).$

2. If $y = \cosh^{-1} x$, then $\cosh y = x$.

$\therefore \quad \frac{1}{2}(e^y + e^{-y}) = x$

i.e., $e^{2y} + 1 = 2x\, e^y$

i.e., $e^{2y} - 2x\, e^y + x^2 = x^2 - 1$

i.e., $e^y - x = \pm \sqrt{x^2 - 1}$

i.e., $e^y = x \pm \sqrt{x^2 - 1}$

$\therefore \quad e^y = x + \sqrt{x^2 - 1}$ or $\dfrac{1}{x + \sqrt{x^2 - 1}}$

$\therefore \quad y = \pm \log_e (x + \sqrt{x^2 - 1})$.

3. If $y = \tanh^{-1} x$, then $x = \tanh y$.

$\therefore \quad \dfrac{e^y - e^{-y}}{e^y + e^{-y}} = x$

i.e., $e^y - e^{-y} = x\, e^y + x\, e^{-y}$

i.e., $e^y (1 - x) = e^{-y} (1 + x)$

i.e., $e^{2y} = \dfrac{1 + x}{1 - x}$.

$\therefore \quad y = \frac{1}{2} \log_e \left(\dfrac{1 + x}{1 - x}\right)$.

§ 15·13. Differentiation of hyperbolic functions.

1. If $y = \sinh x$, we have
$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \tfrac{1}{2}(e^x + e^{-x}) = \cosh x.$$

2. If $y = \cosh x$, we have
$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \tfrac{1}{2}(e^x - e^{-x}) = \sinh x.$$

3. If $y = \tanh x$,
$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)$$
$$= \frac{\cosh x \cdot \frac{d}{dx}(\sinh x) - \sinh x \cdot \frac{d}{dx}(\cosh x)}{\cosh^2 x}$$
$$= \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{\cosh^2 x}$$
$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$
$$= \operatorname{sech}^2 x.$$

4. Similarly, if $y = \coth x$, $\dfrac{dy}{dx} = -\operatorname{cosech}^2 x$.

" if $y = \operatorname{sech} x$, $\dfrac{dy}{dx} = -\dfrac{\sinh x}{\cosh^2 x}$.

" if $y = \operatorname{cosech} x$, $\dfrac{dy}{dx} = -\dfrac{\cosh x}{\sinh^2 x}$.

§ 15·14. Differentiation of the Inverse Hyperbolic functions.

1. If $y = \sinh^{-1} x$, we have $\sinh y = x$.

Differentiating both sides with respect to y,

$$\cosh y = \frac{dx}{dy}.$$

$$\therefore \frac{dx}{dy} = \sqrt{1 + \sinh^2 y}$$

$$= \sqrt{1 + x^2}.$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}.$$

2. If $y = \cosh^{-1} x$, we have $x = \cosh y$.

$$\therefore \frac{dx}{dy} = \sinh y = \sqrt{x^2 - 1}.$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}.$$

3. If $y = \tanh^{-1} x$, we have $\tanh y = x$.

$$\therefore \frac{dx}{dy} = \operatorname{sech}^2 y = 1 - x^2.$$

$$\therefore \frac{dy}{dx} = \frac{1}{1 - x^2}.$$

Similarly, if $y = \coth^{-1} x$, we find $\dfrac{dy}{dx} = -\dfrac{1}{x^2 - 1}$.

Exercises VI.

Differentiate with respect to x the following functions :—

1. $\tan^{-1}(\sqrt{x})$.
2. $\cos^{-1}(2x - 3)$.
3. $\sin^{-1}\left(\dfrac{3x - 1}{4}\right)$.
4. $\cot^{-1}\left(\dfrac{1}{x^2}\right)$.
5. $(\sin^{-1} x)^2$.
6. $x \tan^{-1} x$.
7. $x \sec^{-1} x$.
8. $\log \sin^{-1}(e^x)$.
9. $\sqrt{1 - x^2} \cos^{-1} x$.
10. $\sqrt[3]{\tan^{-1}(e^x)}$.
11. $\sin^{-1}(\cos x)$.
12. $\tan^{-1}(e^{ax})$

13. $\cos^{-1}(1+x^2)^{-1/2}$.
14. $\cot^{-1}(x^3)$.
15. $\cos\left(\sin^{-1}\frac{1}{x}\right)$.
16. $\tan^{-1}(\log x)$.
17. $\sec^{-1}(\log x)$.
18. $\operatorname{cosec}^{-1}(e^{2x+1})$.
19. $\sin^{-1}\left(\frac{b+a\sin x}{a+b\sin x}\right)$.
20. $\sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)$.
21. $\cos^{-1}\left(\frac{b+a\cos x}{a+b\cos x}\right)$.
22. $\cot^{-1}(\operatorname{cosec} 3x)$.
23. $\sinh 2x$.
24. $\operatorname{cosech}\left(\frac{x}{3}\right)$.
25. $\log\sinh(3x)$.
26. $\log\tanh 5x$.
27. $\tan^{-1}\left(\tanh\frac{x}{2}\right)$.
28. $\sinh^{-1}\left(\frac{3x}{4}\right)$.
29. $\cosh^{-1}\left(\frac{2x^2-3}{5}\right)$.
30. $\tanh^{-1}\left(\tan\frac{x}{2}\right)$.
31. $\sinh^{-1}\left(\frac{1-x}{1+x}\right)$.
32. $\tanh^{-1}\sqrt{\tfrac{1}{2}(1+x)}$.
33. Find the value of $\sinh^{-1}\left(\frac{3}{4}\right)$.
34. Show that $\operatorname{sech}^{-1}(\tfrac{1}{2}) = \log(2+\sqrt{3})$.
35. Prove that $\tanh^{-1}\left(\frac{x^2-1}{x^2+1}\right) = \log|x|,\ x \neq 0$.

Differentiate with respect to x:

36. $\sin x\,(\tan^{-1}x)^2$. (B.A. 37 M)
37. $\sin^{-1}[(2x+1)\sqrt{2+x^3}]$. (B.A. 40 M)
38. $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$. (*Hint.* Put $x=\tan\theta$.) (B.Sc. 44 M)
39. $\tan^{-1}(1+\log^2 x)^{1/2}$. (B.A. 44 M)
40. $\sin^{-1}[(2x+1)\sqrt{2-x^3}]$. (B.Sc. 54 S)

§ 16·1. Logarithmic Differentiation.

In the case of a function consisting of a number of factors, it is convenient to take the logarithm before differentiating.

Examples.

Ex. 1. Find the differential coefficient of $\frac{(a-x)^2(b-x)^3}{(c-2x)^3}$.

Let $y = \frac{(a-x)^2(b-x)^3}{(c-2x)^3}$.

Taking logarithms on both sides, we have

$$\log y = 2\log(a-x) + 3\log(b-x) - 3\log(c-2x).$$

Differentiating both sides with respect to x, we have

$$\frac{1}{y}\frac{dy}{dx} = -\frac{2}{a-x} - \frac{3}{b-x} + \frac{6}{c-2x}.$$

$$\therefore \quad \frac{dy}{dx} = y\left(-\frac{2}{a-x} - \frac{3}{b-x} + \frac{6}{c-2x}\right)$$

$$= \frac{(a-x)^2(b-x)^3}{(c-2x)^3}\cdot\left(-\frac{2}{a-x} - \frac{3}{b-x} + \frac{6}{c-2x}\right).$$

Ex. 2. What is the differential coefficient of

$$y = e^{ax}\cos^3 x \,.\, \sin^2 x\ ?$$

Taking logarithms on both sides, we get

$$\log y = ax + 3\log\cos x + 2\log\sin x.$$

$\therefore$ Differentiating with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = a - 3\tan x + 2\cot x.$$

$$\therefore \quad \frac{dy}{dx} = e^{ax}\cos^3 x\sin^2 x\,(a - 3\tan x + 2\cot x).$$

§ 16·2. If the expression to be differentiated contains an index involving x, it is advisable to differentiate after taking logarithms on both sides.

Let $y = u^v$ where u and v are functions of x.

$$\log y = v\log u.$$

Differentiating both sides with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{v}{u}\frac{du}{dx} + \log u\frac{dv}{dx}.$$

Examples.

Ex. 1. Differentiate $(\sin x)^x$. Taking logarithms of both sides, we get

$$\log y = x\log(\sin x).$$

Differentiating both sides, we get

$$\frac{1}{y}\frac{dy}{dx} = \log(\sin x) + x\cot x.$$

$$\therefore \quad \frac{dy}{dx} = (\sin x)^x\{\log(\sin x) + x\cot x\}.$$

Ex. 2. If $y = a^x$, $\log y = x\log_e a$.

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \log_e a.$$

$$\therefore \quad \frac{dy}{dx} = a^x\cdot\log_e a.$$

Ex. 3. If $y = 7^{x^2+2x}$, $\log y = (x^2 + 2x) \log_e 7$.

$$\therefore \frac{1}{y}\frac{dy}{dx} = (2x + 2) \cdot \log_e 7.$$

$$\therefore \frac{dy}{dx} = 2(x + 1) \cdot 7^{x^2+2x} \log_e 7.$$

Exercises VII.

Differentiate the following functions :—

1. $\sqrt{\dfrac{1 + x^2}{1 - x^2}}$.
2. $\sin x . \sin 2x . \sin 3x . \sin 4x$.
3. $\dfrac{x}{(a^2 - x^2)^{3/2}}$.
4. $x e^x \sin x$.
5. x^x.
6. $\dfrac{x^4 \sqrt[3]{x^2 + 4}}{\sqrt{4x^2 - 7}}$.
7. $\dfrac{2x}{(1 + x^2)^{1/2} (1 - x^2)^{3/2}}$.
8. $\dfrac{e^{x^2} \tan^{-1} x}{\sqrt{1 + x^2}}$.
9. $x^{5+\sin x}$.
10. $\sin x . \log x . e^x . (a^2 - x^2)^{3x+7}$.
11. $\dfrac{x^3 \sqrt{2 + 3x}}{(2 + x)(1 - x)}$.
12. $(\log x)^x$.
13. $e^{-x} \sin^n x \cos^m x$.
14. $(\tan x)^{\log x}$.
15. 10^{3x-1}.
16. $(\log x)^{\sin x}$.
17. $\sqrt[3]{\dfrac{x^2 (x^3 - 1)}{(2 - 3x)^5}}$.
18. $(ax + b)^{cx+d}$.
19. $\dfrac{1}{\sqrt[3]{(x - 1)(2x - 1)^2 (4x^2 + 3)^4}}$.
20. $\dfrac{\cos^m x . \sin^n x}{\cosh^2 x}$.
21. $[\tan^{-1} (x^2 + 1)]^{\cos x}$. (B.A. 40 M)
22. $x^{\log \log x}$. (B.Sc. 53 M)
23. $x \log x \, e^{\sin x}$. (B.Sc. 53 M)

§ 16·3. Transformations.

Algebraic or trigonometrical transformations are useful to shorten the work of differentiation.

Examples.

Ex. 1. Differentiate $y = \tan^{-1} \dfrac{2x}{1 - x^2}$.

Put $x = \tan \theta$. $\therefore$ $y = \tan^{-1} \left(\dfrac{2 \tan \theta}{1 - \tan^2 \theta}\right)$

$$= \tan^{-1} (\tan 2\theta) = 2\theta = 2 \tan^{-1} x.$$

$$\therefore \frac{dy}{dx} = \frac{2}{1 + x^2}.$$

Ex. 2. Differentiate $y = \tan^{-1}\dfrac{a-x}{1+ax}$.

Put $a = \tan\alpha$ and $x = \tan\theta$.

$$\therefore\quad y = \tan^{-1}\left(\frac{\tan\alpha - \tan\theta}{1+\tan\alpha\tan\theta}\right)$$
$$= \tan^{-1}\tan(\alpha-\theta)$$
$$= \alpha - \theta$$
$$= \alpha - \tan^{-1}x.$$

Here α is a constant since a is a constant.

$$\therefore\quad \frac{dy}{dx} = -\frac{1}{1+x^2}.$$

Ex. 3. Differentiate $y = \dfrac{x\sqrt{a^2-x^2}}{2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}$.

Put $x = a\sin\theta$.

$$\therefore\quad y = \frac{a\sin\theta\sqrt{a^2-a^2\sin^2\theta}}{2} + \frac{a^2}{2}\cdot\sin^{-1}\left(\frac{a\sin\theta}{a}\right)$$
$$= \frac{a^2}{2}\sin\theta\,.\cos\theta + \frac{a^2}{2}\,.\,\theta$$
$$= \frac{a^2}{4}\,.\sin 2\theta + \frac{a^2}{2}\,.\,\theta.$$
$$\therefore\quad \frac{dy}{dx} = \frac{a^2}{2}\cos 2\theta\frac{d\theta}{dx} + \frac{a^2}{2}\cdot\frac{d\theta}{dx}$$
$$= \frac{a^2}{2}\cdot\frac{d\theta}{dx}(1+\cos 2\theta)$$
$$= a^2\cos^2\theta\,.\frac{d\theta}{dx}$$

$x = a\sin\theta.$

Differentiating with respect to θ, we get

$$\frac{dx}{d\theta} = a\cos\theta.$$
$$\therefore\quad \frac{d\theta}{dx} = \frac{1}{a\cos\theta} = \frac{1}{\sqrt{a^2-x^2}}.$$
$$\therefore\quad \frac{dy}{dx} = a^2\left\{\sqrt{1-\frac{x^2}{a^2}}\right\}^2\cdot\frac{1}{\sqrt{a^2-x^2}} = \sqrt{a^2-x^2}.$$

Exercises VIII.

Differentiate the following functions :—

1. $\cos^{-1}(4x^3 - 3x)$.
2. $\sin^{-1}(3x - 4x^3)$.
3. $\cos^{-1}\sqrt{\dfrac{1+x}{2}}$.
4. $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$.
5. $\tan^{-1}\sqrt{\dfrac{x^2}{a^2-x^2}}$.
6. $\tan^{-1}\left(\dfrac{1-\cos x}{1+\cos x}\right)^{1/2}$.
7. $\tan^{-1}\dfrac{3a^2x - x^3}{a(a^2-3x^2)}$.
8. $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$.
9. $\tan^{-1}(\sqrt{1+x^2} - x)$.
10. $\sin^{-1}(2x\sqrt{1-x^2})$.
11. $\sec^{-1}\left(\dfrac{1}{1-2x^2}\right)$.
12. $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$.
13. $\tan^{-1}\left(\dfrac{\sin x}{1+\cos x}\right)$.
14. $\tan^{-1}\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$.
15. $\sqrt{c^2-x^2} + \dfrac{c}{2}\log\dfrac{c-\sqrt{c^2-x^2}}{c+\sqrt{c^2-x^2}}$.
16. $\tan^{-1}\dfrac{x-\sqrt{a^2-x^2}}{x+\sqrt{a^2-x^2}}$.
17. $\sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right)$.
18. $\sin^{-1}\left(\dfrac{x^2}{\sqrt{x^4+a^4}}\right)$.
19. $\tan^{-1}\left(\dfrac{\sqrt{2ax-x^2}}{a-x}\right)$.
20. $(x-1)\sqrt{2x-x^2} + \sin^{-1}(x-1)$.
21. $\tan^{-1}\dfrac{1}{\sqrt{x^2-1}}$. (B.Sc. 53 M)

§ 17. Differentiation of Implicit functions.

We have considered so far only functions in which the dependent variable is expressed explicitly in terms of the independent variable. If the two variables x and y are connected by a relation of the form $f(x, y) = 0$, it is often difficult or impossible to find y explicitly in terms of x and then obtain its derivative. In such cases, we differentiate the given equation term by term and solve the resulting equation for the differential coefficient.

Examples.

Ex. 1. Find $\dfrac{dy}{dx}$ when x and y are connected by the relation $ax^2 + 2hxy + by^2 = c$.

Differentiating term by term, we get

$$\frac{d}{dx}(ax^2) + \frac{d}{dx}(2hxy) + \frac{d}{dx}(by^2) = \frac{d}{dx}(c)$$

$$i.e., \; 2ax + 2h\left(x\frac{dy}{dx} + y\right) + 2by\frac{dy}{dx} = 0$$

$$i.e., \; (hx + by)\frac{dy}{dx} + (ax + hy) = 0.$$

$$\therefore \quad \frac{dy}{dx} = -\frac{ax + hy}{hx + by}.$$

Ex. 2. If $\sin y = x \sin (a + y)$, prove that

$$\frac{dy}{dx} = \frac{\sin^2 (a + y)}{\sin a}.$$

$$x = \frac{\sin y}{\sin (a + y)}.$$

Differentiating both sides with respect to y,

$$\frac{dx}{dy} = \frac{\sin (a + y) \cos y - \sin y \cos (a + y)}{\sin^2 (a + y)}$$

by § 15·6.

$$= \frac{\sin a}{\sin^2 (a + y)}.$$

Inverting, the result follows.

Ex. 3. If $x (1 + y)^{1/2} + y (1 + x)^{1/2} = 0$, prove that

$$\frac{dy}{dx} = -\frac{1}{(1 + x)^2}. \qquad \text{(B.Sc. Sub. 38)}$$

$$x (1 + y)^{1/2} = -y (1 + x)^{1/2}.$$

Squaring both sides, we get $x^2 (1 + y) = y^2 (1 + x)$

$$i.e., \; x^2 - y^2 + x^2y - xy^2 = 0$$

$$i.e., \; (x - y)(x + y + xy) = 0.$$

$$\therefore \; x = y \text{ or } x + y + xy = 0.$$

If we take $x = y$, we get $2x (1 + x)^{1/2} = 0$, *i.e.*, $x = 0$ or -1.

So we get only trivial values of x and y.

Taking the other relation and differentiating, we get

$$1 + \frac{dy}{dx} + x\frac{dy}{dx} + y = 0.$$

$$\therefore \; \frac{dy}{dx} = -\frac{y + 1}{1 + x}.$$

As $x + y + xy = 0$, $y = -\dfrac{x}{1 + x}$.

Substituting this value of y in $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}.$$

Ex. 4. If $y = \sqrt{(\sin x + \sqrt{\sin x + \sqrt{\sin x}} \ldots\ldots \text{to infinity})}$, find $\frac{dy}{dx}$.

We have $y = \sqrt{\sin x + y}$.

$\therefore \quad y^2 = \sin x + y$.

Differentiating the above equation, we get

$$2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos x}{2y-1}.$$

Ex. 5. If $y = x^{x^{x \ldots \text{to} \infty}}$, find $\frac{dy}{dx}$. (B.Sc. Sub. 45)

$$y = x^{x^{x \ldots \text{to} \infty}} = x^y.$$

Taking logarithms, $\log y = y \log x$.

Differentiating both sides with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}\log x + \frac{y}{x}.$$

$$\therefore \quad x\frac{dy}{dx} = \frac{y^2}{1 - y\log x}.$$

Exercises IX.

Find $\frac{dy}{dx}$ when x and y are connected by the following relations :—

1. $x^2 + y^2 = a^2$.
2. $x^2 + y^2 + 2gx + 2fy + c = 0$.
3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
4. $y^2 = 4ax$.
5. $xy = c^2$.
6. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.
7. $x^3 + y^3 = 3axy$.
8. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.
9. $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$.
10. $27ay^2 = 4(x - 2a)^3$.

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11. If $x^y = y^x$, prove that $\dfrac{dy}{dx} = \dfrac{y(y - x\log y)}{x(x - y\log x)}$.

(B.Sc. Sub. 51 Tr.U.)

12. If $x^y = e^{x-y}$, prove that $\dfrac{dy}{dx} = \dfrac{\log x}{(1+\log x)^2}$.

13. If $y = \dfrac{\sin x}{1+}\,\dfrac{\cos x}{1+}\,\dfrac{\sin x}{1+}\,\dfrac{\cos x}{1+}$ ......, prove that

$$\frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1 + 2y + \cos x \sin x}.$$

14. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, prove that

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}.$$

15. If $y = b\tan^{-1}\left(\dfrac{x}{a} + \tan^{-1}\dfrac{y}{x}\right)$, find $\dfrac{dy}{dx}$. (B.Sc. 51 M)

16. If $y = \sin x^{\sin x^{\sin x \ldots \infty}}$, find $\dfrac{dy}{dx}$.

17. If $(\sin x)^{\cos y} = (\sin y)^{\cos x}$, find $\dfrac{dy}{dx}$.

§ 18. Sometimes both x and y are expressed in terms of a third variable called a parameter. We can always find $\dfrac{dy}{dx}$ in such cases without eliminating the variable.

Let $x = f(t)$ and $y = \phi(t)$.

$$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}.$$

Examples.

Ex. 1. If $x = at^2$, $y = 2at$, find $\dfrac{dy}{dx}$.

$$\frac{dx}{dt} = 2at,\ \frac{dy}{dt} = 2a.$$

$$\therefore\ \frac{dy}{dx} = 2a \div 2at = \frac{1}{t}.$$

Ex. 2. If $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, find $\dfrac{dy}{dx}$.

$$\frac{dx}{d\theta} = a(1 - \cos\theta),\ \frac{dy}{d\theta} = a\sin\theta.$$

$$\therefore \frac{dy}{dx} = a \sin \theta \div a(1 - \cos \theta)$$

$$= \frac{\sin \theta}{1 - \cos \theta}$$

$$= \cot \theta/2.$$

Exercises X.

Find $\frac{dy}{dx}$, when

1. $x = a \cos \theta, y = b \sin \theta.$
2. $x = ct, y = \frac{c}{t}.$
3. $x = a \sec \theta, y = b \tan \theta.$
4. $x = a \cosh \theta, y = b \sinh \theta.$
5. $x = a \cos^3 \theta, y = b \sin^3 \theta.$
6. $x = 2 \sin t, y = \cos 2t.$
7. $x = a\left(\cos \theta + \log \tan \frac{\theta}{2}\right), y = a \sin \theta.$
8. $x = a(\theta + \sin \theta), y = a(1 - \cos \theta).$
9. $x = 3 \cos \theta - \cos^3 \theta, y = 3 \sin \theta - \sin^3 \theta.$
10. $x = a(2 \cos \theta + \cos 2\theta), y = a(2 \sin \theta + \sin 2\theta).$
11. $x = \frac{3at}{1 + t^3}; y = \frac{3at^2}{1 + t^3}.$
12. $x = a \sin 2\theta(1 + \cos 2\theta), y = a \cos 2\theta(1 - \cos 2\theta).$

§ 19. Differentiation of one function with respect to another.

Suppose we have to find differential coefficient of $f(x)$ with respect to $\phi(x)$.

$$\frac{df(x)}{d\phi(x)} = \frac{df(x)}{dz} \text{ where } z = \phi(x)$$

$$= \frac{df(x)}{dx} \cdot \frac{dx}{dz}$$

$$= \frac{d}{dx} f(x) \div \frac{dz}{dx}$$

$$= \frac{d}{dx} f(x) \div \frac{d}{dx} \phi(x).$$

Examples.

Ex. 1. Differentiate $e^{\sin^{-1}x}$ with respect to $\sin^{-1}x$.

$$\frac{de^{\sin^{-1}x}}{d(\sin^{-1}x)} = \frac{d}{dx}(e^{\sin^{-1}x}) \div \frac{d}{dx}(\sin^{-1}x)$$

$$= e^{\sin^{-1}x}\frac{1}{\sqrt{1-x^2}} \div \frac{1}{\sqrt{1-x^2}}$$

$$= e^{\sin^{-1}x}.$$

Ex. 2. Differentiate $\tan^{-1}\frac{2}{1-x^2}$ with respect to $\sin^{-1}\frac{2x}{1+x^2}$.

If we put $x = \tan\theta$,

$$\tan^{-1}\frac{2x}{1-x^2} = 2\theta \text{ and } \sin^{-1}\frac{2x}{1+x^2} = 2\theta.$$

Hence the required derivative is $\dfrac{d(2\theta)}{d(2\theta)} = 1.$

Exercises XI.

1. Differentiate e^t with regard to $\sqrt{t}$. (B.A. 37 M)
2. „ $\sqrt{x+1}$ with respect to x^2. (B.A. 35 M)
3. „ $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$ with respect to $\tan^{-1}x$. (B.A. 52 M)
4. „ $\log_{10}x$ with regard to x^2. (B.A. 51 M)
5. „ $a^{\sin x}$ with respect to $\cos x$. (B.Sc. 33 M)

Miscellaneous Exercises XII.

Find the differential coefficients of

1. $\dfrac{1+x}{\sqrt{1-x}}$.
2. $\sin^{-1}(\cos 2x)$.
3. $\sin^2 x\cos 3x$.
4. $\log(\sec x+\tan x)$.
5. $e^{ax}\log(\sin ax)$.
6. $\log\cot(1-x^2)$.
7. $\log\sec^2(ax+b)$.
8. $\log\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$. ✓
9. $\dfrac{\cos x}{\sin x+\cos x}$. ✓
10. $\sec^{-1}2x$.
11. $\log\tan\dfrac{x}{2}$.
12. $\cos^{-1}\dfrac{x^2+a^2}{x^2+b^2}$. ✓
13. $\log(1+\sqrt{1+\sin x})$.
14. $\sin^{-1}\sqrt{(1+x)}$.

15. $\sec^2(3x+4)$.

16. $e^{\tan^{-1}} \frac{x}{\sqrt{1-x^2}}$.

17. $\log \sin x$.

18. $\log \frac{x^2+x+1}{x^2-x+1}$.

19. $\sin^{-1} \frac{3+4x}{5\sqrt{1+x^2}}$.

20. $\log_a (\tan 2x)$.

21. $\tan^{-1} \frac{a+b\cos x}{b+a\cos x}$.

22. $(\sin x)^{\log x}$.

23. $\frac{x(1+x^2)}{\sqrt{1-x^2}}$.

24. $\tan^{-1} \sqrt{\frac{e^x-1}{e^x+1}}$.

25. $\frac{x-4}{(2x-1)(x-3)}$.

26. $x \tan^{-1} x$.

27. $\log \frac{a+b\cos x}{b+a\cos x}$.

28. $\frac{2x^2-1}{x\sqrt{1+x^2}}$.

29. $\frac{5^x (x+2)^3}{(x+3)^2}$.

30. $\frac{x^3 e^{bx} \sec x}{x^2+1}$.

31. x^{e^x}.

32. $x^x + \sin(\log x)$.

33. $\tan^{-1}\left(\frac{3x+x^3}{1+3x^2}\right)$.

34. $\tan^{-1}\left\{\frac{\sqrt{2ax-x^2}}{a+x}\right\}$.

35. $\frac{(1-x)^{4/3}}{(3x+2)^{1/2}}$.

36. $\log \tan^{-1}\left\{\frac{2x^2+a}{2ax^2-1}\right\}$.

37. $\left(\frac{a}{x}\right)^x$.

38. $\{\tan^{-1}(x^2+1)\}^{\sin x}$.

39. $\frac{1}{(ax+b)^2 \sqrt[3]{(x+d)^4}}$.

40. $\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}$.

41. $\frac{(x+2)^2 \sin x}{\sqrt{2x-1}}$.

42. $x \log x \cdot \log\log x$.

43. $\tan(\sin^{-1} x)$.

44. $\tan^{-1}\sqrt{1+x}$.

45. $\sin^{-1} \frac{x^2+x+4}{(x-2)^3}$.

46. $\log_e \tan(x^2+a^2)$.

47. $x \log_{10} x$.

48. $\tan^{-1}\left(\frac{\cos x}{1+\cos x}\right)$.

49. $x(\log x)^2$.

50. $x^{\cos^{-1} x}$.

51. $(\sin x)^{\cos x}$.

52. $x^3 (x^2+4)^x$.

53. $\sqrt{\frac{1+\sin x}{1-\cos x}}$.

54. $\tan^{-1}\left\{\sqrt{\frac{a-b}{a+b}} \tan\frac{x}{2}\right\}$.

55. $(b^2-x^2)^{3x}$.

56. $\log \frac{x-\sqrt{1-x^2}}{x+\sqrt{1-x^2}}$.

57. $(1+\log x)^{x^2}$.

58. $\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$.

(B.A. Sub. 52 M)

59. $\frac{e^{x^2}}{\sqrt{(x-1)(2-x)}}$.

60. $x(x-2)\sqrt{x+3}$.

a—4

61. $\dfrac{\sin 2x}{e^x}$.

63. $\tan^{-1}\sqrt{\dfrac{1-x^2}{1+x^2}}$. CW (B.A. Sub. 3

62. $\tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$. CW

64. $\cos^{-1}\sqrt{1-x^2}$. (B.A. Sub. 39)

65. $\log(x+\sqrt{x^2+a^2})$.

66. $(a^2+x^2)^x$.

67. $x\sqrt{x^2+a^2}+a^2\log(x+\sqrt{x^2+a^2})$. (B.Sc. Anc. 6

68. $a\log\left(\dfrac{a+\sqrt{a^2-x^2}}{x}\right)-\sqrt{a^2-x^2}$.

69. $\log_{10}(x^2+5x)$. (B.A. 39 S)

70. $\tan^{-1}\left(\dfrac{x\cos a}{1+x\sin a}\right)$. CW

71. $\dfrac{e^{\sin x}}{\sin(x^n)}$.

72. $4^x\log(\sec x^2)$.

73. $10^{\log(\sin x)}$.

74. $(ax^2+bx+c)^{\cos x}$.

75. $\sin^2\left[\cot^{-1}\sqrt{\dfrac{1+x}{1-x}}\right]$. CW

76. Differentiate $\sec^{-1}\dfrac{1}{2x^2-1}$ with respect to $\sqrt{1-x^2}$

77. Differentiate $\tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$ with regard to CW

$$\sec^{-1}\frac{1}{2x^2-1}.$$

78. Find the differential coefficient with respect to x of $\dfrac{a+bx^{3/2}}{x^{5/4}}$ and find the ratio of a to b if this differential coeffic vanishes when $x=5$.

79. $\tan^{-1}\sqrt{\dfrac{1-x^3}{1+x^2}}$.

80. Show that if $y=ax^{nx}$, where a and n are const $\dfrac{dy}{dx}=ny(1+\log x)$.

81. If $y=x\tan\frac{1}{2}x$, prove that $\dfrac{dy}{dx}=\dfrac{x+\sin x}{1+\cos x}$.

82. Differentiate, with respect to x, $\operatorname{cosec}^{-1}\left(\dfrac{1+x^2}{2x}\right)$.

83. If $u = x^y$, find $\frac{du}{dx}$ where y is a function of x.

84. If $y = x\sqrt{\frac{1+x^2}{1-x^2}}$, find the value of $\frac{dy}{dx}$ in terms of x.

85. Find the derivative with respect to x of the function $\frac{x^4+22x^2+9}{x^3+3x}$ and show that the derivative vanishes when $x^2 = 1$ and for two other values of x^2.

86. If $y = \log\frac{x}{2^x}$, find the value of x for which $\frac{dy}{dx} = 0$.

87. $x^m y^n = (x+y)^{m+n}$, prove that $x\frac{dy}{dx} = y$.

88. Differentiate with respect to x,

$$\log\left(\frac{1+x}{1-x}\right)^{1/4} - \frac{1}{2}\tan^{-1}x.$$

89. If $y = e^{x+3x^2}$, prove that $\frac{dy}{dx} = (1+6x)\,y$.

90. If $y = \tan^{-1}\left(\frac{x}{1+x^2}\right) + \tan^{-1}\left(\frac{1+x^2}{x}\right)$, prove that $\frac{dy}{dx} = 0$.

91. Given that

$$\sin\theta\sin(2a+\theta)\sin(4a+\theta)\ldots\ldots\sin\{2(n-1)a+\theta\} = \frac{\sin n\theta}{2^{n-1}} \text{ where } 2na = \pi;$$

prove that

(i) $\cot\theta + \cot(2a+\theta) + \cot(4a+\theta) + \ldots\ldots + \cot\{2(n-1)a+\theta\} = n\,.\cot n\theta.$

(ii) $\operatorname{cosec}^2\theta + \operatorname{cosec}^2(2a+\theta) + \operatorname{cosec}^2(4a+\theta)\ldots\ldots + \operatorname{cosec}^2\{2(n-1)a+\theta\} = n^2\operatorname{cosec}^2 n\theta.$

92. Given that $\cos\frac{x}{2}\,.\cos\frac{x}{2^2}\,.\cos\frac{x}{2^3}\ldots\ldots\infty = \frac{\sin x}{x}$;

prove that

(i) $\frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \frac{1}{2^3}\tan\frac{x}{2^3} + \ldots\ldots\infty = \frac{1}{x} - \cot x.$

(ii) $\frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{2^2} + \frac{1}{2^6}\sec^2\frac{x}{2^3} + \ldots\ldots\infty$

$$= \operatorname{cosec}^2 x - \frac{1}{x^2}.$$

93. Prove that if $x < 1$

(i) $\dfrac{1}{1+x} + \dfrac{2x}{1+x^2} + \dfrac{4x^3}{1+x^4} + \dfrac{8x^7}{1+x^8} + \ldots\ldots \infty = \dfrac{1}{1-x}$.

(ii) $\dfrac{1-2x}{1-x+x^2} + \dfrac{2x-4x^3}{1-x^2+x^4} + \dfrac{4x^3-8x^7}{1-x^4+x^8} + \ldots\ldots \infty$

$$= \frac{1+2x}{1+x+x^2}.$$

94. Differentiate

(i) $\left(1+\log\dfrac{x}{n}\right)\left(\dfrac{x}{n}\right)^{nx}$.

(ii) $\log\dfrac{b+a\cos x+\sqrt{b^2+a^2}\sin x}{a+b\cos x}$.

(iii) $x^{\log \tan x}$. (B.Sc. 52 Tr. U.)

95. Find $\dfrac{dy}{dx}$, when (i) $y = x^{\sin x} + (\sin x)^x$.

(ii) $(x^2+y^2)^2 = 4a(x^3+y^3)$.

(iii) $(\sin x)^y = x + y$. (An. U. 45 Eng.)

(iv) $x\sqrt{y} + y\sqrt{x} = \sqrt{x} - \sqrt{y}$.

(B.Sc. Anc. 62)

96. If $x = e^{\tan^{-1}\frac{y-x^2}{x^2}}$, find $\dfrac{dy}{dx}$. (B.Sc. 50 M)

97. If $y = b\tan^{-1}\left\{\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}\right\}$, find $\dfrac{dy}{dx}$.

98. If $y = x^{yx}$, find $\dfrac{dy}{dx}$. (B.A. 49 M)

99. If $(\cos x)^y = (\sin y)^x$, find $\dfrac{dy}{dx}$. (B.A. 49 M)

100. If $y = x\log\dfrac{y}{a+bx}$, find $\dfrac{dy}{dx}$. (B.Sc. 53 M)

101 If $y = \sin(x+y)^2$, find $\dfrac{dy}{dx}$. (B.A. 54 M)

102. $\sin^{-1}(\sin^{-1} x)$.

Differentiate w.r. to x (103 to 110) :

103. $\tan^{-1}\dfrac{(x+\sqrt{x})}{1-x\sqrt{x}}$. (Tr. U. 55)

104. $(1+\log x)^{x^x}$. (Tr. U. 55)

105. $[\log(\sec x + \tan x)]^{\cot x}$. (Tr. U. 55)

106. $\sin\dfrac{x}{2}\sin\dfrac{x^2}{2^2}\sin\dfrac{x^3}{2^3}\ldots\sin\dfrac{x^n}{2^n}$. (Tr. U. 55)

107. $\log\left(e^{x/2} + \sqrt{e^x + e^{-x}}\right)$. (Tr. U. 55)

108. $\log(\cos e^{x^n})$. (B.A. 55 M)

109. $\log\left(\dfrac{\sqrt{x+a} + \sqrt{x-a}}{\sqrt{x-b} - \sqrt{x-c}}\right)$. (B.A. 56 M)

110. $x + x^{1/x}$. (B.Sc. 55 M)

111. If $x = 1 + t^3$ and $y = 1 + t^2$, show that
$$\frac{d^2y}{dx^2} \Big/ \left(\frac{dy}{dx}\right)^4 \text{ is a constant.}$$

112. Differentiate $\log\{\sqrt{\operatorname{cosec} x + 1} - \sqrt{\operatorname{cosec} x - 1}\}$ w.r. to x.

113. Differentiate $\tan^{-1}\dfrac{x - \sqrt{x}}{1 + x^{3/2}}$ w.r. to x. (B.Sc. 66)

CHAPTER III

SUCCESSIVE DIFFERENTIATION.

§ 20·1. We have seen that the derivative of a function of x is (in general) also a function of x. The new functions may be differentiable, in which case, the derivative of the first derivative is called the second derivative of the original function. Similarly the derivative of the second derivative is called the third derivative; and so on up to the nth derivative.

Thus if $y = 4x^5$,

$$\frac{dy}{dx} = 20x^4$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 80x^3$$

$$\frac{d}{dx}\left\{\frac{d}{dx}\left(\frac{dy}{dx}\right)\right\} = 240x^2, \text{ etc.}$$

The symbols of the successive derivatives are usually abbreviated as follows :—

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D^2y.$$

$$\frac{d}{dx}\left\{\frac{d}{dx}\left(\frac{dy}{dx}\right)\right\} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = D^3y.$$

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = D^ny.$$

If $y = f(x)$, the successive derivatives are also denoted by

$f'(x), f''(x), \ldots\ldots f^n(x)$,

$y', y'', \ldots\ldots y^{(n)}$,

$y_1, y_2, \ldots\ldots y_n$.

§ 20·2. The nth derivative.

For certain functions a general expression involving n may be found for the nth derivative. The usual plan is to find a number of successive derivatives, as many as may be necessary to discover their law of formation and then by induction write down the nth derivative.

For example, if $y = e^{ax}$,

$$\frac{dy}{dx} = a\,e^{ax};$$

$$\frac{d^2y}{dx^2} = a^2\,e^{ax}.$$

Then we can write $\dfrac{d^ny}{dx^n} = a^n\,e^{ax}$.

§ 20·3. Standard results.

1. If $y = (ax + b)^m$, then

$$y_1 = m \cdot a\,(ax + b)^{m-1}.$$
$$y_2 = m\,(m - 1)\,a^2\,(ax + b)^{m-2}.$$
$$y_3 = m\,(m - 1)\,(m - 2)\,a^3\,(ax + b)^{m-3}.$$

Hence $y_n = m\,(m - 1) \ldots (m - n + 1)\,a^n\,(ax + b)^{m-n}$.

In particular, $D^n\,(ax + b)^{-1} = (-1)^n\,n\,!\,a^n\,(ax + b)^{-n-1}$.

2. If $y = \log\,(ax + b)$, then

$$y_1 = a\,(ax + b)^{-1}.$$
$$y_n = a\frac{d^{n-1}}{dx^{n-1}}\,(ax + b)^{-1}$$
$$= a\,(-1)^{n-1}\,(n - 1)\,!\,a^{n-1}\,(ax + b)^{-n}$$
$$= (-1)^{n-1}\,(n - 1)\,!\,a^n\,(ax + b)^{-n}.$$

3. If $y = \sin\,(ax + b)$, then

$$y_1 = a\cos\,(ax + b) = a\sin\left(\frac{\pi}{2} + ax + b\right).$$

Thus the effect of a differentiation is to multiply by a and increase the angle by $\pi/2$.

$$y_2 = a^2\cos\left(\frac{\pi}{2} + ax + b\right) = a^2\sin\left(2 \times \frac{\pi}{2} + ax + b\right)$$

$$y_3 = a^3\sin\left(3\,\frac{\pi}{2} + ax + b\right).$$

In general, $D^n \sin\,(ax + b) = a^n\sin\left(\dfrac{n\pi}{2} + ax + b\right)$.

4. Similarly $D^n\cos\,(ax + b) = a^n\cos\left(\dfrac{n\pi}{2} + ax + b\right)$.

Corollaries : Putting $a = 1$ and $b = 0$,

$$D^n\,(\sin x) = \sin\left(\frac{n\pi}{2} + x\right);\quad D^n\,(\cos x) = \cos\left(\frac{n\pi}{2} + x\right).$$

5. If $y = e^{ax}\sin\,(bx + c)$, then

$$y_1 = e^{ax}\,b\cos\,(bx + c) + a\,e^{ax}\sin\,(bx + c).$$

Putting $a = r\cos\phi$ and $b = r\sin\phi$, we have

$$y_1 = r\,e^{ax}\sin(bx + c + \phi).$$

Thus the effect of a differentiation is to multiply by r and increase the angle by ϕ.

Similarly $y_2 = r^2 e^{ax}\sin(bx + c + 2\phi)$ ......

In general,

$$D^n\{e^{ax}\sin(bx + c)\} = r^n e^{ax}\sin(bx + c + n\phi)$$

where $r = (a^2 + b^2)^{1/2}$ and $\phi = \tan^{-1}\left(\dfrac{b}{a}\right)$.

6. Similarly

$$D^n\{e^{ax}\cos(bx + c)\} = r^n e^{ax}\cos(bx + c + n\phi)$$

where r and ϕ have the same meanings as before.

§ 20·4. Fractional expressions of the form $\dfrac{f(x)}{\phi(x)}$, both functions being algebraic and rational, can be differentiated n times by splitting them into partial fractions.

Examples.

Ex. 1. Find y_n where $y = \dfrac{3}{(x+1)(2x-1)}$.

Resolving into partial fractions, we obtain

$$y = \frac{2}{2x-1} - \frac{1}{x+1}.$$

$$\therefore\ y_n = \frac{2(-1)^n 2^n . n!}{(2x-1)^{n+1}} - \frac{(-1)^n n!}{(x+1)^{n+1}}$$

$$= (-1)^n n!\left\{\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{1}{(x+1)^{n+1}}\right\}.$$

Ex. 2. Find y_n when $y = \dfrac{x^2}{(x-1)^2(x+2)}$.

Let $\dfrac{x^2}{(x-1)^2(x+2)} \equiv \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+2}$.

Then we easily find that $A = 5/9$, $B = 1/3$ and $C = 4/9$.

$$\therefore\ y = \frac{5}{9}\frac{1}{x-1} + \frac{1}{3}\frac{1}{(x-1)^2} + \frac{4}{9}\frac{1}{(x+2)}.$$

Hence $y_n = \dfrac{5}{9}\dfrac{n!(-1)^n}{(x-1)^{n+1}} + \dfrac{(n+1)!(-1)^n}{3(x-1)^{n+2}} + \dfrac{4}{9}\dfrac{(-1)^n n!}{(x+2)^{n+1}}$

$$= (-1)^n n!\left\{\frac{5}{9(x-1)^{n+1}} + \frac{n+1}{3(x-1)^{n+2}} + \frac{4}{9(x+2)^{n+1}}\right\}$$

Ex. 3. Find y_n when $y = \dfrac{1}{x^2 + a^2}$. (B.Sc. Sub. 54)

$$y = \frac{1}{x^2 + a^2} = \frac{1}{2ai}\left[\frac{1}{x - ai} - \frac{1}{x + ai}\right]$$

$$\therefore\ y_n = \frac{(-1)^n\, n\,!}{2ai}\left[\frac{1}{(x - ai)^{n+1}} - \frac{1}{(x + ai)^{n+1}}\right].$$

§ 20·5. Trigonometrical transformation.

It is possible to break up products of powers of sines and cosines into a sum by Trigonometrical methods.

Examples.

Ex. 1. Find the nth differential coefficient of

$$\cos x \,.\, \cos 2x \,.\, \cos 3x.$$

$$\begin{aligned}\cos x \cos 2x \cos 3x &= \tfrac{1}{2} \cos 2x (\cos 4x + \cos 2x)\\ &= \tfrac{1}{2} \cos 2x \cos 4x + \tfrac{1}{2} \cos^2 2x\\ &= \tfrac{1}{4} (\cos 2x + \cos 6x) + \tfrac{1}{4} (1 + \cos 4x)\\ &= \tfrac{1}{4} + \tfrac{1}{4} (\cos 2x + \cos 4x + \cos 6x).\end{aligned}$$

$$\therefore\ D^n (\cos x \cos 2x \cos 3x)$$

$$= \frac{1}{4}\left\{2^n \cos\left(\frac{n\pi}{2} + 2x\right) + 4^n \cos\left(\frac{n\pi}{2} + 4x\right) + 6^n \cos\left(\frac{n\pi}{2} + 6x\right)\right\}.$$

Ex. 2. Find the nth differential coefficient of

$$\cos^5 \theta \sin^7 \theta.$$

Let $x = \cos \theta + i \sin \theta$; then $\dfrac{1}{x} = \cos \theta - i \sin \theta$.

$$\therefore\ x + \frac{1}{x} = 2 \cos \theta\ ;\ x - \frac{1}{x} = 2i \sin \theta.$$

Also by De Moivre's Theorem, we have

$$x^n = \cos n\theta + i \sin n\theta\ ;\ \frac{1}{x^n} = \cos n\theta - i \sin n\theta$$

so that $x^n + \dfrac{1}{x^n} = 2 \cos n\theta$ and $x^n - \dfrac{1}{x^n} = 2i \sin n\theta$.

We have $2^5 \cos^5 \theta = \left(x + \dfrac{1}{x}\right)^5$ and

$$2^7 i^7 \sin^7 \theta = \left(x - \frac{1}{x}\right)^7.$$

Hence $2^{12} i^7 \cos^5\theta \sin^7\theta = \left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^7$

$$= \left(x^2-\frac{1}{x^2}\right)^5\left(x-\frac{1}{x}\right)^2$$

$$= \left(x^{10}-5x^6+10x^2-\frac{10}{x^2}+\frac{5}{x^6}-\frac{1}{x^{10}}\right)\left(x^2-2+\frac{1}{x^2}\right)$$

$$= \left(x^{12}-\frac{1}{x^{12}}\right)-2\left(x^{10}-\frac{1}{x^{10}}\right)-4\left(x^8-\frac{1}{x^8}\right)$$
$$+10\left(x^6-\frac{1}{x^6}\right)+5\left(x^4-\frac{1}{x^4}\right)-20\left(x^2-\frac{1}{x^2}\right)$$

Hence we have

$-2^{11}\cos^5\theta\sin^7\theta =$
$\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta.$

$$D^n(\cos^5\theta\sin^7\theta) = -1/2^{11}\left\{12^n\sin\left(\frac{n\pi}{2}+12\theta\right)\right.$$
$$-10^n.2\sin\left(\frac{n\pi}{2}+10\theta\right)-8^n.4\sin\left(\frac{n\pi}{2}+8\theta\right)$$
$$+6^n.10\sin\left(\frac{n\pi}{2}+6\theta\right)+4^n.5\sin\left(\frac{n\pi}{2}+4\theta\right)$$
$$\left.-2^n.20\sin\left(\frac{n\pi}{2}+2\theta\right)\right\}.$$

Exercises XIII.

(1) Find the nth differential coefficient of

1. $\sin^3 x$.
2. $\cos^4 x$.
3. $\sin^3 x\cos^5 x$.
4. $\sin^2 x\cos^3 x$. (B.A. 50 M)
5. $\sin x.\sin 2x.\sin 3x$.
6. $e^x\sin x$.
7. $e^{4x}\sin^2 x$.
8. $e^x\sin x\sin 2x$.
9. $\frac{ax+b}{cx+d}$.
10. $e^{5x}\sin^3 ax$.
11. $\frac{1}{4x^2+8x+3}$.
12. $\frac{x^2}{(x+1)^2(x+2)}$. (B.A. 49 M)
13. $\frac{x^4}{(x-1)(x-2)}$.
14. $\log(4-x^2)$.
15. $\frac{1}{x^2-a^2}$.
16. $\frac{1}{4x^2-1}$. (B.Sc. 43 M)
17. $\frac{x^2}{(x-a)(x-b)(x-c)}$.
18. $\frac{x^3}{(x-a)(x-b)(x-c)}$. (B.Sc. 55 M)

(2) Prove that if $y^3 - 3ax^2 + x^3 = 0$,

$$\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0.$$ (B.Sc. 51 M)

(3) If $x = a(t - \sin t)$, $y = a(1 + \cos t)$, find $\frac{d^2y}{dx^2}$ as a function of t. (B.A. 41 M)

(4) If $y = e^{\frac{x}{\sqrt{2}}} \cos \frac{x}{\sqrt{2}}$, show that

$$\frac{d^3y}{dx^3} = e^{\frac{x}{\sqrt{2}}}\left(\frac{x}{\sqrt{2}} + \frac{3\pi}{4}\right).$$ (B.A. 46 M)

(5) If $ax^2 + 2hxy + by^2 = 1$, show that $D^2y = \frac{h^2 - ab}{(hx + by)^3}$.

(6) If $x^3 + y^3 - 3axy = 0$, prove that $D^2y = \frac{2a^3 xy}{(ax - y^2)^3}$.

(7) Prove that if $y = \frac{ax + b}{cx + d}$, then $2y_1 y_3 = 3y_2^2$ and that if $a + d = 0$, $(y - x) y_2 = 2y_1(1 + y_1)$. (B.A. 38 M)

(8) If $y^2 = (x - a)(x - b)$, show that

$$\frac{d^3}{dx^3}\left[\left(\frac{d^2y}{dx^2}\right)^{-2/3}\right] = 0.$$

(9) Find y_n when (i) $y = \tan^{-1}\frac{x}{a}$.

(ii) $y = \frac{1}{(x + a)^2 + b^2}$.

(iii) $y = \frac{1}{(x^2 + a^2)(x^2 + b^2)}$.

(iv) $y = \frac{x}{(x - 1)^2 (x + 2)}$. (B.Sc. 62 M)

§ 20·6. Formation of equations involving derivatives.

When a relation between x and y is given, we can in many cases deduce from it a relation between the variables x, y and the derivatives of y with respect to x as the following examples will show :

Examples.

Ex. 1. If $xy = ae^x + be^{-x}$, prove that

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0.$$

Here $xy = ae^x + be^{-x}$.

Now differentiating both sides with respect to x, we have

$$y + x\frac{dy}{dx} = ae^x - be^{-x}.$$

Differentiating both sides of the equation once again, we get

$$\frac{dy}{dx} + x\frac{d^2y}{dx^2} + \frac{dy}{dx} = ae^x + be^{-x}$$

i.e., $$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy$$

i.e., $$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0.$$

Ex. 2. Prove that if $y = \sin(m\sin^{-1}x)$,

$$(1-x^2)y_2 - xy_1 + m^2y = 0.$$

$$y = \sin(m\sin^{-1}x).$$

$$\therefore \sin^{-1}y = m\sin^{-1}x.$$

Differentiating both sides with respect to x, we get

$$\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = \frac{m}{\sqrt{1-x^2}}.$$

Squaring and simplifying, we have

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2(1-y^2).$$

Differentiating the above equation with respect to x, we get

$$(1-x^2)\,2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = -2m^2y\frac{dy}{dx}.$$

Cancelling the common factor $2\frac{dy}{dx}$ throughout, we get

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0.$$

Ex. 3. If $x = \sin\theta$, $y = \cos p\theta$, prove that

$$(1-x^2)y_2 - xy_1 + p^2y = 0.$$

$$x = \sin\theta,\ y = \cos p\theta.$$

$$\therefore \frac{dx}{d\theta} = \cos\theta,\ \frac{dy}{d\theta} = -p\sin p\theta.$$

$$\therefore \frac{dy}{dx} = -p\cdot\frac{\sin p\theta}{\cos\theta} = -p\cdot\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}.$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = p^2\frac{1-y^2}{1-x^2}.$$

$$\therefore (1-x^2)\left(\frac{dy}{dx}\right)^2 = p^2(1-y^2).$$

Differentiating the above equation and cancelling the common factor $2\frac{dy}{dx}$, we get

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+p^2y=0.$$

Exercises XIV.

1. If $xy=ax^2+\frac{b}{x}$, prove that

$$x^2\frac{d^2y}{dx^2}+2\left(x\frac{dy}{dx}-y\right)=0.$$ (B.A. Sub. 40)

2. If $y=ax\cos mx$, prove that

$$x^2\left(\frac{d^2y}{dx^2}+m^2y\right)=2\left(x\frac{dy}{dx}-y\right).$$ (B.A. Sub. 42)

3. If $y=e^{-x}\cos x$, prove that

$$\frac{d^4y}{dx^4}+4y=0.$$ (B.A. Sub. 38)

4. If $y=x^2\cos x$, prove that

$$x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(x^2+6)y=0.$$ (B.A. Sub. 37)

5. If $y=(x+\sqrt{1+x^2})^m$, prove that

$$(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-m^2y=0.$$ (B.A. 39 M)

6. If $y=(\sqrt{x}-1)e^{\sqrt{x}}$, show that

$$2x\frac{d^2y}{dx^2}=\frac{y}{2}+\frac{dy}{dx}.$$

7. If $y\sqrt{x}=\sin^{-1}\sqrt{x}$, show that

$$(4x-4x^2)\frac{d^2y}{dx^2}+(6-8x)\frac{dy}{dx}=y.$$ (B.Sc. 39 M)

8. If $y=(A+Bx)e^{kx}$, show that

$$\frac{d^2y}{dx^2}-2k\frac{dy}{dx}+k^2y=0.$$ (B.Sc. 35 M)

9. If $y=(\sin^{-1}x)^2$, show that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-2=0.$$ (B.Sc. 33 M ; B.A. 51 M)

10. If $y=a\cos(\log x)+b\sin(\log x)$, show that

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0.$$ (B.A. Sub. 50)

11. If $x = \sin t$, $y = \sin pt$, prove that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0.$$ (B.Sc. 39 M

12. If $y = Ae^{-kt}\cos(pt + e)$, show that

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + n^2y = 0, \text{ where } n^2 = p^2 + k^2.$$

13. If $y = e^x \tan^{-1} x$, prove that

$$(1 + x^2)y_2 - 2(1 - x + x^2)y_1 + (1 - x)^2y = 0.$$

14. If $y = Ae^x + Be^{x^2}$, show that

$$(2x - 1)\frac{d^2y}{dx^2} - (4x^2 + 1)\frac{dy}{dx} + 2(2x^2 - x + 1)y = 0$$

15. If $y = A\{x + \sqrt{x^2 - 1}\}^n + B\{x - \sqrt{x^2 - 1}\}^n$, prove that $(x^2 - 1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - n^2y = 0.$ (B.Sc. 38 M

16. If $y = e^{-2x}\cos 3x$, find constants a and b such that, for all values of x, $\dfrac{d^2y}{dx^2} + a\dfrac{dy}{dx} + by = 0.$

17. If $y = \dfrac{ax^2 + bx + c}{1 - x}$, show that $(1 - x)y_3 = 3y_2$.

§ 21·1. Leibnitz formula for the nth derivative of a product.

This formula expresses the nth derivative of the product of two variables in terms of the variables themselves and their successive derivatives.

If u and v are functions of x, we have

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

i.e., $$D(uv) = v\,Du + u\,Dv.$$

Differentiating again with respect to x, we get

$$D^2(uv) = D(v \,.\, Du) + D(u \,.\, Dv)$$
$$= v\,D^2u + 2Du \,.\, Dv + u \,.\, D^2v.$$

Similarly $D^3(uv) = v.D^3u + 3D^2u.Dv + 3Du.D^2v + u.D^3v.$

However far this process may be continued, it will be seen that the numerical coefficients follow the same law as that of the Binomial Theorem and the indices of the derivatives correspond to the exponents of the Binomial Theorem. Hence

$$\frac{d^n}{dx^n}(uv) = \frac{d^nu}{dx^n}v + {}^nC_1\frac{d^{n-1}u}{dx^{n-1}}\cdot\frac{dv}{dx} + {}^nC_2\frac{d^{n-2}u}{dx^{n-2}}\cdot\frac{d^2v}{dx^2}$$
$$+ \dots\dots + {}^nC_r\cdot\frac{d^{n-r}u}{dx^{n-r}}\cdot\frac{d^rv}{dx^r} + \dots\dots + {}^nC_1\frac{du}{dx}\cdot\frac{d^{n-1}v}{dx^{n-1}} + u\cdot\frac{d^nv}{dx^n}.$$

§ 21·2. A complete formal proof by induction may be given as follows.

Assume the theorem to be true for some one value of n, *i.e.*, suppose $D^n(uv) = u_n v + {}_nC_1\, u_{n-1} v_1 + {}_nC_2\, u_{n-2} v_2 + \cdots$
$$\cdots + {}_nC_{r-1}\, u_{n-r+1} v_{r-1} + {}_nC_r\, u_{n-r} v_r + \cdots + u v_n.$$

Differentiating again, we get

$$D^{n+1}(uv) = (u_{n+1} v + u_n v_1) + {}_nC_1 (u_n v_1 + u_{n-1} v_2) +$$
$${}_nC_2 (u_{n-1} v_2 + u_{n-2} v_3) + \cdots {}_nC_{r-1} (u_{n-r+2} v_{r-1} + u_{n-r+1} v_r)$$
$$+ {}_nC_r (u_{n-r+1} v_r + u_{n-r} v_{r+1}) + \cdots + \cdots + (u_1 v_n + u v_{n+1})$$
$$= u_{n+1} v + (1 + {}_nC_1) u_n v_1 + ({}_nC_1 + {}_nC_2) u_{n-1} v_2 + \cdots$$
$$\cdots + ({}_nC_{r-1} + {}_nC_r) u_{n-r+1} v_r + \cdots + u v_{n+1}.$$

Now ${}_nC_{r-1} + {}_nC_r = {}_{(n+1)}C_r$ and so

$$1 + {}_nC_1 = {}_{(n+1)}C_1$$
$${}_nC_1 + {}_nC_2 = {}_{(n+1)}C_2$$
$${}_nC_2 + {}_nC_3 = {}_{(n+1)}C_3 \text{ and so on.}$$

$$\therefore \quad D^{n+1}(uv) = u_{n+1} v + {}_{(n+1)}C_1\, u_n v_1 + {}_{(n+1)}C_2\, u_{n-1} v_2$$
$$+ \cdots + {}_{(n+1)}C_r\, u_{n-r+1} v_r + \cdots + u v_{n+1}.$$

Hence if the theorem be true for any value of n, it must be true for the next higher value $n + 1$. It has been seen that it is true for $n = 1$ and therefore it is true for $n = 2$ and therefore for $n = 3$ and so on for all values of n.

This theorem is particularly useful when one of the two factors is a small integral multiple of x; if this be taken as v in the preceding formula, its differential coefficients and the series will consist of only a few terms.

Examples.

Ex. 1. Find the nth differential coefficient of $x^2 \log x$.

Taking $v = x^2$ and $u = \log x$,

$$\frac{d^n}{dx^n}(x^2 \log x) = \frac{d^n}{dx^n}(\log x) \cdot x^2 + {}_nC_1 \cdot \frac{d^{n-1}}{dx^{n-1}}(\log x) \frac{d}{dx}(x^2)$$
$$+ {}_nC_2 \frac{d^{n-2}}{dx^{n-2}}(\log x) \frac{d^2}{dx^2}(x^2).$$

All the other terms will be zero since the successive derivatives of x^2 after the second derivative vanish.

$$\therefore \quad D^n(x^2 \log x) = \frac{(-1)^{n-1}(n-1)!}{x^n} x^2 +$$
$$n \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} 2x + \frac{n(n-1)}{2} \frac{(-1)^{n-3}(n-3)!\,2}{x^{n-2}}$$
$$= \frac{(-1)^{n-3} \cdot 2 \cdot (n-3)!}{x^{n-2}}.$$

Ex. 2. If $y = \sin(m \sin^{-1} x)$, prove that

$$(1 - x^2) y_2 - xy_1 + m^2 y = 0 \text{ and}$$
$$(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} + (m^2 - n^2) y_n = 0.$$

(B.A. 49 ; 52)

(*Vide* example 2 in § 20·6.)

$$(1 - x^2) y_2 = xy_1 - m^2 y.$$

Taking the nth derivative of each term by Leibnitz's Theorem, we have

$$y_{n+2}(1 - x^2) + {}_nc_1 y_{n+1}(-2x) + {}_nc_2 y_n(-2)$$
$$= y_{n+1} x + {}_nc_1 y_n - m^2 y_n$$

i.e., $y_{n+2}(1 - x^2) - 2nx y_{n+1} - n(n-1) y_n$

$$= xy_{n+1} + ny_n - m^2 y_n$$

i.e., $(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} + (m^2 - n^2) y_n = 0$

Exercises XV.

1. Find the nth differential coefficients of

(1) $x e^x$. (2) $x^2 e^{3x}$.
(3) $x \sin x$. (4) $x^2 \cos x$.
(5) $e^x \log x$. (6) $x^3 \sin^3 x$. (B.Sc. 52 M)
(7) $x^n a^x$. (B.Sc. 53 M)
(8) $x^2 \sin 3x$. (B.Sc. Comp. 62)

2. If $y = x^2 e^x$, show that

$$y_n = \tfrac{1}{2} n(n-1) y_2 - n(n-2) y_1 + \tfrac{1}{2}(n-1)(n-2) y,$$

where y_n stands for $\dfrac{d^n y}{dx^n}$. (B.A. Sub. 50)

3. If $y = \dfrac{\sinh^{-1} x}{\sqrt{1 + x^2}}$, prove that

$$(1 + x^2) y_{n+2} + (2n + 3) xy_{n+1} + (n + 1)^2 y_n = 0.$$

(B.Sc. 50 M)

4. If $I_n = \dfrac{d^n}{dx^n}(x^n \log x)$, prove that

$$I_n = n!\left\{\log x + 1 + \frac{1}{2} + \frac{1}{3} + \dots\dots + \frac{1}{n}\right\}.$$

(B.Sc. 51 M)

5. If $y = \sin^{-1} x$, prove that

$$(1 - x^2) y_2 - xy_1 = 0 \text{ and}$$
$$(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} - n^2 y_n = 0.$$

(B.Sc. 51 M)

6. If $y = e^{a \sin^{-1} x}$, prove that

$$(1 - x^2) y_2 - xy_1 - a^2 y = 0.$$

Hence show that

$$(1 - x^2) y_{n+2} - (2n + 1) x y_{n+1} - (n^2 + a^2) y_n = 0.$$

(B.A. 46 M ; B.A. Sub. 52)

7. If $y = a \cos(\log x) + b \sin(\log x)$, show that

$$x^2 y_{n+2} + (2n + 1) x y_{n+1} + (n^2 + 1) y_n = 0.$$

(B.Sc. 47 ; B.Sc. 50 M)

8. If $y^{1/m} + y^{-1/m} = 2x$, prove that

$$(x^2 - 1) y_{n+2} + (2n + 1) x y_{n+1} + (n^2 - m^2) y_n = 0.$$

9. If $\cos^{-1}\left(\dfrac{y}{b}\right) = n \log\left(\dfrac{x}{n}\right)$, prove that

$$x^2 y_{n+2} + (2n + 1) x y_{n+1} + 2n^2 y_n = 0.$$

10. If $x + y = 1$, prove that

$$\frac{d^n}{dx^n}(x^n y^n) = n!\{y^n - ({}_nc_1)^2 y^{n-1} x + ({}_nc_2)^2 y^{n-2} x^2 + \ldots$$
$$\ldots + (-1)^n x^n\}.$$

11. If $y = \dfrac{\log x}{x^2}$, show that

$$x^3 \frac{d^3y}{dx^3} + 6x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} - 4y = 0.$$

Using Leibnitz's theorem, differentiate this three times.

(B.Sc., Sub. 47)

12. If $y = x^{n-1} \log x$, prove that $xy_1 = (n - 1) y + x^{n-1}$. By differentiating this result $(n - 1)$ times, show that

$$y_n = \frac{(n-1)!}{x}. \qquad \text{(B.Sc. 48 M)}$$

13. If $y = (x + \sqrt{1 + x^2})^m$, prove that

$$(1 + x^2) y_{n+2} + (2n + 1) x y_{n+1} + (n^2 - m^2) y_n = 0.$$

(B.Sc. 49 M) [*vide* Ex. 5, Ex. XIV.]

14. Prove that

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = (-1)^n \frac{n!}{x^{n+1}} \left\{\log x - 1 - \frac{1}{2} - \frac{1}{3} \ldots - \frac{1}{n}\right\}.$$

(B.Sc. 53 M)

15. If $U = \dfrac{x^5}{Ax^2 + 2Bx + C}$, show that

$$(Ax^2 + 2Bx + C) U_n + 2n (Ax + B) U_{n-1}$$
$$+ n(n-1) A U_{n-2} = 0 \text{ if } n > 5.$$

(B.A. 44 M)

16. If $y = (1 - x)^{-a} e^{-ax}$, prove that $(1 - x) \dfrac{dy}{dx} = axy$ and that $(1 - x) y_{n+1} - (n + ax) y_n - nay_{n-1} = 0$.

(B.Sc. Sub. 54)

CHAPTER IV

MEANING OF THE DERIVATIVE

§ 22. Till now, we have explained the various methods of determining the differential coefficients of functions. We shall give in this chapter the various interpretations for the derivative and some of its applications.

§ 23·1. Geometrical interpretation.

Let us assume that P and Q are two neighbouring points on the continuous curve $y = f(x)$. From P and Q draw PM, QN perpendicular to the x-axis and from P draw PR perpendicular to QN.

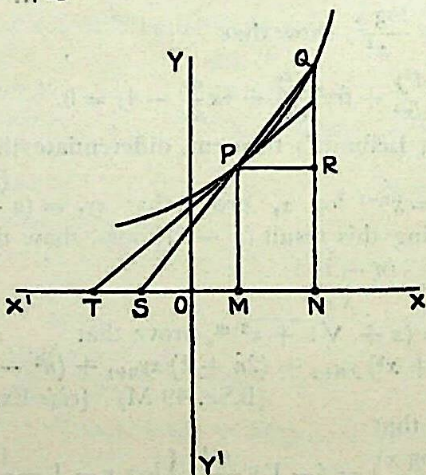

Fig. 4

If P remains fixed and Q moves towards P along the curve and finally coincides with P, then the chord PQ becomes the tangent at P.

The inclination of the tangent at P to OX is $\angle PTX$.

$$\text{Here } \angle PTX = \underset{Q \to P}{\text{Lt}} \angle PSX$$

$$= \underset{Q \to P}{\text{Lt}} \angle QPR.$$

Let the coordinates of P, Q be (x, y) and $(x + \Delta x, y + \Delta y)$ respectively.

Then $\tan \angle QPR = \dfrac{RQ}{PR} = \dfrac{\Delta y}{\Delta x}$.

$$\tan \angle PTQ = \operatorname*{Lt}_{Q \to P} \tan \angle QPR = \operatorname*{Lt}_{Q \to P} \frac{\Delta y}{\Delta x}$$

$$= \operatorname*{Lt}_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

Hence $\dfrac{dy}{dx}$ is the gradient of the tangent to the curve at the point (x, y).

§ 23·2. Meaning of the sign of the differential coefficient.

Let $y = f(x)$ be a function of x. When $x = a$, the function has the value $f(a)$. If x is given a small increment (positive), $f(x)$ may be greater or less than $f(a)$. If the value of y is greater than $f(a)$, then $f(x)$ is called an *increasing* function at $x = a$. If the value of y is less than $f(x)$, then $f(x)$ is called a *decreasing* function at $x = a$.

$$\frac{dy}{dx} \text{ at } x = a \text{ is } \operatorname*{Lt}_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

If $\dfrac{dy}{dx}$ is positive, then $f(a+h) - f(a)$ is positive provided h is small. Therefore, if h is positive, $f(a+h) - f(a)$ is positive and, if h is negative, $f(a+h) - f(a)$ is negative.

If h is small and positive, we get

$$f(a-h) < f(a) < f(a+h).$$

So, at $x = a$, if $\dfrac{dy}{dx}$ is positive, it means that $f(x)$ increases with x at $x = a$. On the other hand if $\dfrac{dy}{dx}$ is negative at $x = a$, it means that $f(x)$ decreases as x increases.

Examples.

Ex. 1. Prove that the function $f(x) = x^3 - 3x^2 + 6$ is positive for all values of $x > 2$. (B.A. 39 M)

$$f'(x) = 3x^2 - 6x$$
$$= 3x(x-2).$$

$\therefore$ If $x > 2$, then $f'(x)$ is positive.

$\therefore$ $f(x)$ is an increasing function for $x > 2$ but

$f(2) = 2$, *i.e.*, $+$ve.

Therefore for $x > 2$, $f(x)$ is positive.

Ex. 2. For what values of x is $2x^3 - 9x^2 + 12x +$ [illegible] a decreasing function ?

$$f(x) = 2x^3 - 9x^2 + 12x + 4.$$

Then $f'(x) = 6x^2 - 18x + 12$
$= 6(x-2)(x-1).$

If the value of x lies between 1 and 2, $f'(x)$ is negative and so in that range $f(x)$ is a decreasing function.

Ex. 3. Show that for $x > 0$, $x - \frac{1}{2}x^2 < \log(1+x) < x$.

Let $f(x)$ be $x - \frac{1}{2}x^2 - \log(1+x)$.

$$\therefore \quad f'(x) = 1 - x - \frac{1}{1+x}$$

$$= -\frac{x^2}{1+x}.$$

If x is positive, $f'(x)$ is negative.

$\therefore$ $f(x)$ is a decreasing function.

The value of $f(x)$ at $x = 0$ is 0.

$\therefore$ For all positive values of x, $f(x)$ is negative.

$\therefore$ $x - \frac{1}{2}x^2 - \log(1+x) < 0.$

$\therefore$ $x - \frac{1}{2}x^2 < \log(1+x).$

Let $F(x)$ be $\log(1+x) - x$.

$$F'(x) = \frac{1}{1+x} - 1$$

$$= -\frac{x}{1+x}.$$

If x is positive, $F'(x)$ is negative.

$\therefore$ $F(x)$ is a decreasing function.

$F(0) = 0$. So if $x > 0$, $F(x)$ is negative.

$\therefore$ $\log(1+x) - x < 0$

i.e., $\log(1+x) < x$.

Hence $x - \frac{1}{2}x^2 < \log(1+x) < x$.

Exercises XVI.

1. Show that the polynomials

$$2x^3 + 3x^2 - 12x + 7$$
$$3x^4 + 8x^3 - 6x^2 - 24x + 19$$

are positive when $x > 1$.

2. Find the range of values of x for which the function $x^3 - 6x^2 - 36x + 7$ is increasing with x. (B.A. 37 M.)

3. For what values of x is $2x^3 - 15x^2 - 84x +$ [illegible] a decreasing function ?

4. Show that when x is positive,

$$1-\frac{x^2}{2}<\cos x<1\ ;$$

$$x-\frac{x^3}{6}<\sin x<x\ ;$$

$$x>\log(1+x)>\frac{x}{1+x}\ ;$$

$$e^x>1+x+\frac{x^2}{2}.$$ (B.Sc. 45 M)

$$\tan^{-1}x>\frac{x}{1+x^2}\ ;$$

$$(x-1)\,e^x+1>0.$$ (B.Sc. 52 M)

5. Without using infinite series, prove that when x is positive, $\log(1+x)$ lies between $x-\frac{1}{2}x^2$ and $x-\frac{1}{2}x^2+\frac{1}{3}x^3$.

(B.Sc. 38 M)

6. Prove that $x\sin x+\cos x+\frac{1}{2}\cos^2 x$ is a decreasing function of x in $\left(0,\frac{\pi}{2}\right)$.

7. Prove that $f(x)=\frac{1}{2}\sin x\tan x-\log\sec x$ is a positive and increasing function of x in $0<x<\pi/2$.

8. Prove that $\dfrac{1-2x-x^2}{1+x-2x^2}$ always decreases as x increases.

9. Show that $\dfrac{\log x}{x}$ steadily decreases as x increases from e upwards.

10. Prove that $\log(1+x)$ lies between

$$x-\frac{x^2}{2} \text{ and } x-\frac{x^2}{2(1+x)},$$ where x is positive.

11. Show that $\dfrac{\log(1+x)}{x}$ and $\dfrac{x}{(1+x)\log(1+x)}$ both decrease steadily as x increases from zero to infinity.

12. Show that $x-\sin x$ is an increasing function for all values of x. Determine for what values of a, $ax-\sin x$ is a steadily increasing or decreasing function of x. (B.Sc. 39 M)

13. Find the condition to be satisfied by the coefficients of x^3+ax^2+bx+c for the function to be an increasing function for all values of x. (B.A. 54 M)

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14. Find for what values of a the function $x^3 - ax^2 + 48x + 1$ will increase, as x increases, for all values of x. (Tr. U. 55)

15. Prove that for $0 < x < \pi/2$, $\tan x > x$, $\sec^2 x > 1 + x^2$ and $\tan x > x + x^3/3$. (B.Sc. 55 M)

16. Show that $\dfrac{a \sin x + b \cos x}{a \cos x - b \sin x}$ always increases as x increases.

(B.A. Sub. 55 M)

§ 24. Rate of change of variables.

The rate of change of variables can be measured by differential coefficients.

Let $y = f(x)$. Then the values of y corresponding to a and $a+h$ of x are $f(a)$ and $f(a+h)$.

Change in x is h. Corresponding to this change of x, the change in y is $f(a+h) - f(a)$.

$\therefore$ In the range $(a, a+h)$, the average rate of change of y with respect to $x = \dfrac{f(a+h) - f(a)}{h}$.

Here, when $h \to 0$, the range $(a, a+h)$ becomes the point $x = a$.

$\therefore$ At $x = a$, the rate of change of y with respect to x

$$= \underset{h \to 0}{\text{Lt}} \frac{f(a+h) - f(a)}{h} = f'(a).$$

Generally the change of y with respect to x is $\dfrac{dy}{dx}$.

Examples.

Ex. 1. At the point (2, 5) on the curve, $y = x^3 - 2x + 1$, show that the gradient is increasing 12 times as fast as x.

(B.A. Sub. 35)

$$y = x^3 - 2x + 1.$$

$$\text{Gradient} = \frac{dy}{dx} = 3x^2 - 2.$$

The rate of change of $\dfrac{dy}{dx}$ with respect to x

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right) = 6x.$$

$\therefore$ At the point (2, 5) the value of $\dfrac{d^2y}{dx^2} = 12.$

$\therefore$ The rate of change of the gradient with respect to x is 12.

Ex. 2. The radius of a circular plate is increasing in length at ·01 inch per second. What is the rate at which the area is increasing when the radius is 12 inches ? (B.A. Sub. 38)

Let the radius of the circular plate be r inches at time t seconds.

$\therefore$ The area of the circular plate at time $t = \pi r^2$.

The rate of change of radius with respect to time is given as ·01. $\therefore \dfrac{dr}{dt} = \cdot 01$.

Here we have to find the rate of change of the area with respect to t, *i.e.*, $\dfrac{dA}{dt}$.

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi r \times \cdot 01.$$

$$\therefore \quad \frac{dA}{dt} \text{ when } r = 12 \text{ inches} = 2\pi \times 12 \times \cdot 01$$

$$= \cdot 24\pi \text{ sq. inch per second.}$$

Ex. 3. Water is dripping out at the steady rate of 1 c.c. per second through a tiny hole at the vertex of a conical vessel whose axis is vertical. When the slant height of the water in the filter is 4 cm., find the rate of decrease of (1) the slant height of water, (2) the area of the water surface, given that the vertical angle of the vessel is 60°.

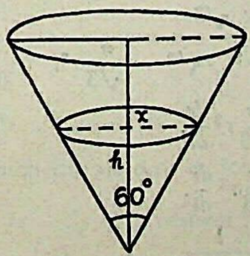

Fig. 5

Let the radius of the water surface, depth of water and volume of the water in the cone at time t sec. be r cm., h cm. and V c.c. respectively.

$\therefore \quad V = \frac{1}{3}\pi r^2 h.$

Let the slant height at that time be l.

Then $\quad l^2 = r^2 + h^2.$

Since the semi-vertical angle is 30°, $r = l/2$ and $h = \sqrt{3}\, l/2$.

$$\therefore \quad V = \frac{1}{3}\pi \times \left(\frac{l}{2}\right)^2 \times \frac{\sqrt{3}}{2}\, l = \frac{\pi}{8\sqrt{3}}\, l^3.$$

Differentiating V with respect to t, we get

$$\frac{dV}{dt} = \frac{3\pi}{8\sqrt{3}} \cdot l^2 \frac{dl}{dt}.$$

It is given that water is dripping out at the rate of 1 c.c. per second.

$$i.e., \quad \frac{dV}{dt} = -1.$$

$$\therefore \quad -1 = \frac{3\pi}{8\sqrt{3}}\, l^2 \frac{dl}{dt}.$$

$$\therefore \quad \frac{dl}{dt} = -\frac{8\sqrt{3}}{3\pi l^2}.$$

When $l = 4$ cm., $\dfrac{dl}{dt} = -\dfrac{8\sqrt{3}}{3\pi \times 16} = -\dfrac{1}{2\sqrt{3}\pi}$ cm. per sec.

We can express A, the area of the water surface, in terms of the radius r.

$$A = \pi r^2.$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

When $l = 4$ cm., $r = 2$ cm.

$\tan 30° = r/h$.

$$\therefore \quad h = r \times \sqrt{3}.$$

$$\therefore \quad V = \tfrac{1}{3}\pi r^2 \cdot r\sqrt{3} = \frac{\pi}{\sqrt{3}}\, r^3.$$

$$\frac{dV}{dt} = \frac{3\pi}{\sqrt{3}}\, r^2 \frac{dr}{dt}.$$

$$-1 = \frac{3\pi}{\sqrt{3}}\, r^2 \frac{dr}{dt}.$$

$$\therefore \frac{dr}{dt} = -\frac{1}{\sqrt{3}\pi r^2}.$$

$$\therefore \frac{dA}{dt} = -2\pi r \frac{1}{\sqrt{3}\pi r^2}$$

$$= -\frac{2}{\sqrt{3} \cdot r}.$$

The rate of decrease of the surface area when $r = 2$ is $\dfrac{1}{\sqrt{3}}$ sq. cm./sec.

Ex. 4. A man 6 feet high walks at a uniform rate of 4 miles per hour away from a lamp 20 feet high. Find the rate at which the length of his shadow increases. (B.A. Sub. 50)

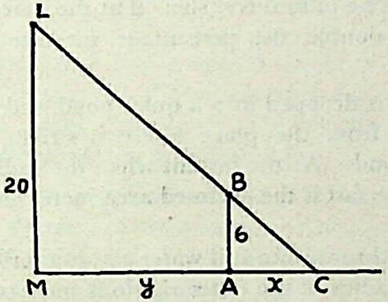

Fig. 6

Let L be the lamp and AB the position of the man at t seconds. The length of his shadow is AC and let it be x feet. The rate at which he walks is given, *i.e.*, the rate at which AM $=y$ is changing is given.

i.e., $\frac{dy}{dt}=4$ miles per hour

$=88/15$ feet per second.

ΔABC and ΔLMC are similar.

$$\therefore \quad \frac{CA}{CM}=\frac{6}{20}$$

i.e., $\frac{x}{x+y}=\frac{3}{10}$

i.e., $\quad 7x=3y.$

The rate at which the shadow is increasing is $\frac{dx}{dt}$.

Differentiating $7x=3y$ with respect to t, we get

$$7\frac{dx}{dt}=3\frac{dy}{dt}.$$

$\frac{dy}{dt}$ is given as $\frac{88}{15}$ feet per second.

$$\therefore \quad \frac{dx}{dt}=\frac{3\times 88}{15\times 7}\text{ feet per second}$$

$$=\frac{88}{35}\text{ feet per second.}$$

Exercises XVII.

1. If the rate of increase of $x^3 - 5x^2 + 5x + 8$ is twice the rate of increase of x, what are the values of x ?

2. For a circle of radius r, show that the percentage increase in the area is double the percentage increase in r for small increments. (B.A. Sub. 35)

3. A stone is dropped into a quiet pond and waves move in circles outward from the place where it strikes, at a speed of 3 inches per second. At the instant when the radius of the wave ring is 3 feet, how fast is the enclosed area increasing ? (B.A. 37 M)

4. A stone thrown into still water causes a series of concentric ripples. If the radius of the outer ripple is increasing at the rate of 5 feet per second, how fast is the area of the disturbed water increasing when the outer ripple has a radius of 12 feet ? (B.A. Sub. 52)

5. A balloon which aiways remains spherical is being inflated by pumping in 10 cubic inches of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15 inches. (B.A. Sub. 39)

6. In the isothermal expansion of a gas, given $pv = c$, for what value of p will the rate of change of pressure per unit change of volume be double ? What was it when p was 20 ? (B.A. Sub. 46)

7. The adiabatic law for the expansion of air is $PV^{1\cdot4} = C$, where C is constant. If at a given time the volume V is observed to be 10 cubic feet and the pressure P is 50 lbs. per square inch, at what rate is the pressure changing at that instant, if the volume is decreasing at one cubic foot per second ? (B.Sc. 32 M)

8. The top of the ladder 20 feet long is resting against a vertical wall on a level pavement, when the ladder begins to slide outward. At the moment when the foot of the ladder is 12 feet from the wall, it is sliding away from the wall at the rate of 2 feet per second. How fast is the top sliding downwards at this instant? How far is the foot from the wall when it and the top are moving at the same rate ? (B.Sc. 37 M)

9. A rod 13 feet long slides with its ends A, B on two straight lines at right angles which meet at O. If the end A moves away from O with uniform speed of 4 feet per second, find the speed of the end B when A is 5 feet from O. (B.A. Sub. 45)

10. A ladder 20 feet long has one end on the ground and the other in contact with a vertical wall. The lower end slips along the ground. Show that when the foot of the ladder is 16 feet away from the wall, the upper end is moving $\frac{4}{3}$ times as fast as the lower end. (B.A. 35 M)

11. The ends A and B of a line 20 inches in length move on two lines at right angles. If the velocity of B is 8 inches per second when 12 inches from O, find the velocity of A; also find the velocity of the middle point of the line. (B.Sc. 35 M)

12. An inverted cone has a depth of 10 cms. and a base of radius 5 cms. Water is poured into it at the rate of $1\frac{1}{2}$ c.c. per minute. Find the rate at which the level of the water in the cone is rising when the depth is 4 cms. (B.A. 52 M)

13. A hollow right circular cone of height 10 feet and semi-vertical angle 30° is full of water. The water is drawn off at a variable rate such that the height of water decreases at a uniform rate of one inch per second.

Find the rate at which (1) the volume, (2) the area of the free surface of the water is diminishing at the instant when the height of the water is 6 feet. (B.Sc. 39 M)

14. A filter paper is in the form of a circular cone, the radius of the base being 2 inches and the altitude 4 inches. If water flows out of the filter at a constant rate of 2·25 cubic inches per minute, find the rate at which the level of the water falls when the depth of the water is 3 inches. (B.A. Sub. 41)

15. A conical funnel of radius 2 inches and of the same depth is filled with a solution which filters at the rate of one cubic inch per minute. How fast is the surface falling when it is one inch from the top of the funnel? (B.Sc. 33 M)

16. Sand is being poured on the ground from the orifice of an elevated pipe and forms a pile which has always the shape of a right circular cone whose height is equal to the radius of the base. If the sand is falling at the rate of 6 c. ft. per second, find the rate at which the height of the pile is rising when the height is 5 feet. (B.A. Sub. 33)

17. A hemispherical bowl, radius 12 inches, is being filled with water at the rate of $\frac{3}{4}$ cubic foot per minute. Find the rate at which the depth of the water is increasing when the water is 8 inches deep. (B.A. 39 M)

18. A hemispherical bowl of radius 3 feet is full of water with its rim horizontal. If a pipe at the bottom empties it at the rate of 2 cubic feet per minute, find at what rate the water level begins to descend. (B.A. Sub. 47)

19. The section of a canal is a right-angled isosceles triangle with the right angle below and the hypotenuse horizontal. A portion of the canal 24 feet long is blocked by planks at the ends perpendicular to its length and water is allowed to flow into it at the rate of 360 cubic feet per hour. Find the rate at which the level of the water rises when the depth is 3 feet. (B.A. Sub. 33)

20. A man 6 feet tall walks directly away at the rate of 3 miles per hour from a point source of light at the top of a pole 20 feet high. How fast does the end of his shadow move when he is 50 feet away from the pole ? (B.A. Sub. 45)

21. An arc light is hung 15 feet directly above the straight horizontal wall at which a man 6 feet tall is walking. How fast is the man's shadow lengthening when he is walking at the rate of 175 feet per minute ? (B.A. Sub. 35)

22. A circular disc of area 10 sq. ins. is distant $2\frac{1}{2}$ feet from a wall and parallel to the wall. A point source of light is moved in a straight line passing through the centre of the disc and perpendicular to it at the rate of 5 feet per second. Find the rate of growth of the area of the shadow of the disc on the wall when the light is 20 feet from the wall. (B.A. 21 M)

23. A candle is moving directly away from the centre of a sphere of radius 5 feet at the rate of 2 feet per second. At what rate is the surface illuminated by the candle increasing when the candle is 10 feet from the centre of the sphere ? (B.A. Sub. 32)

24. A stone falls freely under gravity being dropped from a point distant 10 feet and at the same level as a lamp whose height above the ground is 30 ft. Find the velocity of the shadow of the stone on the ground one second after it is dropped taking the acceleration due to gravity as 32 ft./sec.^{2} (T.U. B.Sc. 41)

25. Two men P and Q are walking along two roads OA and OB at right angles to one another with uniform speeds of 4 and 3 miles per hour respectively. P passes O one hour earlier than Q. Determine the rate at which the distance between them increases when Q is 6 miles beyond O. (B.A. 50 M)

26. A kite flying at a height of h feet has x feet of string paid out at time t seconds. If the kite moves horizontally with

a velocity of v feet per second, find at what rate the string is being paid out. (Assume the string is straight.)

27. A point P moves on the curve $y = 2x^2$ at 4 feet per second. At what rate is the inclination of the tangent to the curve at P changing when P passes through (1, 2) ? (B.A. 30 M)

28. A circle is drawn with its centre on a given parabola and touching its axis. Show that if the point of contact recedes with a constant velocity from the vertex of the parabola the rate of increase of the area of the circle is also constant. (B.A. Sub. 36)

29. A is a fixed point on the circumference of a circle whose centre is O and radius one foot. A point P, starting from A, describes the circle uniformly in one second. Find the rate at which the area of a triangle AOP is changing at the instant when the angle AOP is 60°. (B.A. 38 M)

30. A military observer in an aeroplane is ascending at the rate of a miles an hour. If the radius of the earth be r miles, how fast is the visible area of the earth's surface increasing in square miles per minute, t minutes after the plane left the ground? (B.Sc. 48 M)

§ 25. Velocity and Acceleration.

If a particle moves along a straight line OA, so that the time t seconds after passing O it is at P where OP is s feet, the motion is known completely if s is given as a function of t.

Suppose $s = f(t)$.

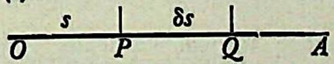

Fig. 7

If the particle is at Q after $(t + \Delta t)$ seconds, we have

$$OQ = s + \Delta s = f(t + \Delta t).$$

$$\therefore \quad PQ = \Delta s = f(t + \Delta t) - f(t).$$

Now the average velocity from P to Q is $\dfrac{\Delta s}{\Delta t}$ ft. per sec. and as Q tends to P, this average velocity approaches in value the velocity at P.

$\therefore$ The velocity at P, in ft. per second, equals

$$\underset{\Delta t \to 0}{\text{Lt}} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

and is calculated from the relation

$$\frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = f'(t).$$

$\therefore$ If the velocity of the particle after t seconds is v ft. per second and if $s = f(t)$,

$$v = \frac{ds}{dt} = f'(t).$$

The same method gives an expression for the rate at which the velocity is increasing and this is called the *acceleration* of the particle. If the velocity is decreasing, the acceleration is negative.

If the velocity after t seconds is v ft. per second where v is a known function of t, the velocity after $t + \Delta t$ seconds may be denoted by $v + \Delta v$ ft. per second.

$\therefore$ The velocity increases by Δv ft. per second in Δt seconds.

$\therefore$ The average rate at which the velocity is increasing when the particle moves from P to Q, is

$$\frac{\Delta v}{\Delta t} \text{ ft. per second per second.}$$

But as Q tends to P, *i.e.*, when Δt becomes smaller and smaller, the more nearly the average acceleration from P to Q approaches in value the acceleration at P.

$\therefore$ The acceleration at P in ft. per second per second equals

$$\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}.$$

$\therefore$ If the acceleration of the particle after t seconds is a ft. per second per second, we have

$$a = \frac{dv}{dt}.$$

Examples.

Ex. 1. A stone thrown vertically upwards rises s ft. in t seconds where $s = 80t - 16t^2$. What is its velocity after 2 seconds ? Find its acceleration.

$$s = 80t - 16t^2.$$

$$\frac{ds}{dt} = 80 - 32t.$$

$\therefore$ Velocity after t seconds $= 80 - 32t$.

$\therefore$ Velocity after 2 seconds $= 80 - 32 \times 2$ ft. per second

$= 16$ ft. per second.

$$\text{Acceleration} = \frac{dv}{dt} = -32 \text{ ft./sec.}^2$$

Ex. 2. Prove that

$$\frac{dv}{dt} = v\frac{dv}{ds}.$$

If $v^2 = s^2 + 4s + 4$, find the acceleration 10 ft. away from its starting point.

$$\frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt}.$$

$$\text{But } \frac{ds}{dt} = v. \quad \therefore \quad \frac{dv}{dt} = \frac{dv}{ds} v.$$

$$v^2 = s^2 + 4s + 4.$$

Differentiating this with respect to s, we get

$$2v \frac{dv}{ds} = 2s + 4.$$

If $s = 10$, $v^2 = 10^2 + 4 \times 10 + 4 = 144$.

$\therefore \quad v = 12$ ft./sec.

Substituting these values of s and v in

$$2v \frac{dv}{ds} = 2s + 4,$$

we get $\frac{dv}{ds} = 1$.

Acceleration $= v \cdot \frac{dv}{ds} = 12$ ft./sec.^{2} where $s = 10$.

Ex. 3. A particle moves along a straight line OA so that t seconds after passing O, it is s feet from O, where $s = 10 + 27t - t^3$. Discuss the motion of the particle given $t = 0$, $s = 10$.

$\therefore$ The particle starts from a point 10 feet from O.

Velocity after t seconds $= \frac{ds}{dt} = 27 - 3t^2$.

$\therefore$ Initial velocity, *i.e.*, the velocity at start is got by putting $t = 0$.

Initial velocity $= 27$ feet per second.

If $t = \pm 3$, $\frac{ds}{dt} = 0$.

Negative value for time t is meaningless. When $t = 3$, $s = 64$ feet and when $t > 3$, $\frac{ds}{dt}$ is $-$ve.

$\therefore$ After 3 seconds, the direction of velocity changes and the particle returns towards O.

When the particle is at O, we get

$$10 + 27t - t^3 = 0.$$

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The time taken to return to O can be got by solving the above equation.

$$\text{Acceleration} = \frac{dv}{dt} = -6t.$$

Exercises XVIII.

1. Find the velocity and acceleration in the following cases :—

(1) $s = 12 + 20t - 2t^2$.
(2) $s = t^3 - 2t^2 + 4$.
(3) $s = 10 \cos \frac{\pi t}{2}$.
(4) $s = 6t + \frac{7}{t+1}$.

2. Find the initial velocity and the distance of starting in the following cases :—

(1) $s = t^4 + 3t + 4$.
(2) $s = 8 \cos 2t + 4 \sin t$.

3. A particle moves along a st. line OA so that t seconds after passing O, it is s feet from O, where $s = 6t^2 - \frac{1}{2}t^3$. Find (1) its velocity and acceleration 3 seconds and 8 seconds after passing O; (2) the time at which it is momentarily at rest. (3) its greatest velocity in the direction OA.

4. A stone thrown vertically upwards from the top of a tower 64 feet high rises s feet in t seconds, where $s = 48t - 16t^2$. Find,

(1) its upward velocity after 1 second, 2 seconds and at start ;

(2) the time at which it is momentarily at rest and its greatest height above the ground ;

(3) its velocity when it strikes the ground.

5. The space-time formula for the motion of a particle along a straight line is $s = t^3 - 9t^2 + 24t - 18$. Prove that its velocity is zero for two values of t. Find its velocity when the acceleration is zero and the value of the acceleration when the velocity is zero.

6. The distance described by a particle in t seconds is given by $s = ae^t + be^{-t}$. Show that the acceleration is always equal to the distance passed over. (B.A. Sub. 35)

7. At the end of t seconds, the distances of a moving point from two rectangular axes are given by $x = a + c \cos t$, $y = b + c \sin t$. Show that the resultant velocity and acceleration

are constants in magnitude. Find the path of the moving point. (B.A. Sub. 34)

8. A body moves in a straight line in such a manner that $s = \frac{1}{2} vt$, s being the space travelled in time t and v the velocity at the end of time t. Prove that the acceleration is constant. (B.A. Sub. 46)

9. A point P is moving with uniform velocity V along a straight line AB. O is a point on a perpendicular to AB at A and at a distance l from it. Show that the angular velocity of P about O is $\frac{lV}{OP^2}$. (B.A. 40 M)

10. A particle moves along a straight line so that its distance x from a fixed point on the line at time t is given by

$$t = \sqrt{2ax - x^2} + 2a \cos^{-1} \sqrt{\frac{x}{2a}}.$$

Find the speed and the acceleration of the particle as functions of x. (B.A. 41 M)

11. The velocity v of a point moving along a straight line, when at a distance of x from the origin, is given by $a + bv^2 = x^2$. Show that the acceleration of the particle is x/b. (B.A. 41 M)

12. A particle moves along the curve $y = f(x)$ so that its velocity parallel to the x-axis is constant. Find its velocity parallel to the y-axis in any position.

A particle moves freely along a parabolic path under gravity. A light is placed at a point O on its path, and the particle casts its shadow on a vertical line in its plane of flight. Show that the shadow moves vertically down with a constant velocity. (B.A. 42 M)

13. A lamp post is 30 feet high and it is distant 10 feet from a tower 200 feet high. A stone is dropped from the top of the tower and strikes the ground. Find the velocity of the shadow of the stone just before it strikes the ground. (B.Sc. 57)

14. A particle is moving along a straight line so that at time t its displacement s from a fixed point in the line is given by $s = a \sin (pt + b)$ where a, p, b are constants. If v and f are respectively the velocity and acceleration of the particle at time t, prove that (i) $v^2 - fs$ is constant, (ii) $\frac{df}{ds}$ is constant, and (iii) $s \frac{df}{dt} = vf$.

15. A particle describes an ellipse whose semi-axes are 4 feet and 3 feet with a constant speed of 1 foot per second. Find the velocity of the foot of the perpendicular from the particle on the major axis when the particle is at a distance 1 foot from the major axis. (B.Sc. 66)

CHAPTER V

ROLLE'S THEOREM AND MEAN VALUE THEOREMS

§ 26·1. Rolle's Theorem.

If $f(x)$ is continuous in the closed* interval $a \leqslant x \leqslant b$ an
if $f'(x)$ exists in the open interval $a < x < b$ and if $f(x)$ is zer
when $x = a$ and when $x = b$, then $f'(x)$ will be zero for at lea
one value of x between a and b.

This theorem is obvious at once from a figure.

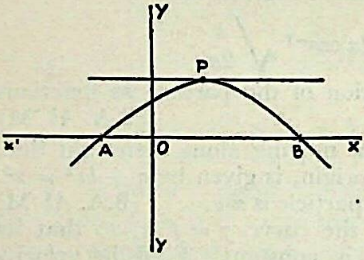

Fig. 8 (*a*)

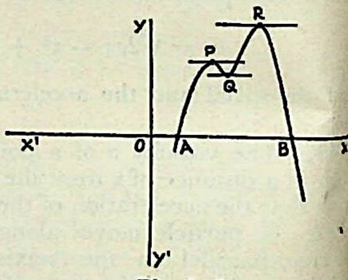

Fig. 8 (*b*)

Suppose the curve $y = f(x)$ cuts the x-axis at A ($x = a, y = 0$) and B ($x = b$, $y = 0$) ; then it is obvious that, if the curv $y = f(x)$ be continuous between A and B and possesses a uniqu tangent at every point between A and B, the tangent line must b parallel to the axis of x at an odd number of intermediate poin such as P.

Hence at P, the gradient of the curve, *i.e.*, $f'(x) = 0$.

Proof : When $f(x)$ satisfies the above conditions, tw possibilities arise :

(1) Either $f(x) = 0$ throughout $a \leqslant x \leqslant b$ in which cas $f'(x) = 0$ at all points in the interval.

(2) Or $f(x) \neq 0$ at some point of the interval (a, b). $f(x)$ is given to be continuous in (a, b) and vanishes at both $x =$ and $x = b$. By a property of continuous functions, it attains maximum or minimum at an odd number of points in the interva at each of which points $f'(x)$ vanishes.

* A closed interval includes the end points while an open interv excludes them.

The same proposition is also enunciated as follows :—

The real root of the equation $f'(x) = 0$ lies between every adjacent two of the real roots of the equation $f(x) = 0$.

Cor. Rolle's theorem can be extended to the case when $f(x) = k$ at $x = a$ and $x = b, k \neq 0$. Taking $F(x) = f(x) - k$ and applying the previous result to $F(x)$, $F'(x) = 0$ at some point in (a, b), *i.e.*, $f'(x) = 0$ at some point in (a, b).

Exercises XIX.

1. Verify the theorem when $f(x) = (x - a)^m (x - b)^n$. The function vanishes for $x = a$ and $x = b$.

$$f'(x) = (x - a)^{m-1} (x - b)^{n-1} \{ (m + n) x - mb - na \}$$

$f'(x)$ vanishes for $x = \dfrac{mb + na}{m + n}$ which lies between a and b.

2. Show that if a rational integral function of x vanishes for n values between given limits, its first differential coefficient will vanish for at least $(n - 1)$ values of x respectively between the same limits.

3. Prove that, if $\dfrac{a_0}{n+1} + \dfrac{a_1}{n} + \ldots\ldots + \dfrac{a_{n-1}}{2} + a_n = 0$, then the equation $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_n = 0$ has at least one root between 0 and 1.

§ 26·2. Mean Value Theorem.

If $f(x)$ is continuous in the closed interval $a \leq x \leq b$ and $f'(x)$ exists in the open interval $a < x < b$, then there is at least one value of x, say x_1, between a and b such that

$$f'(x_1) = \frac{f(b) - f(a)}{b - a}.$$

Consider the function $F(x)$ given by

$$F(x) = f(b) - f(x) - (b - x) R \tag{1}$$

where R is a constant given by the equation $F(a) = 0$;

$$i.e., \quad f(b) - f(a) - (b - a) R = 0;$$

$$i.e., \quad R = \frac{f(b) - f(a)}{b - a}.$$

$$F(b) = f(b) - f(b) - (b - b) R = 0.$$

$$F(a) = 0.$$

$F(x)$ is a continuous function of x and $F'(x)$ exists as $f'(x)$ exists.

$\therefore$ By Rolle's Theorem, $F'(x)$ will be zero for at least one value of x, say, x_1 between a and b.

$$i.e., \quad F'(x_1) = 0.$$

Differentiating equation (1), we get

$$F'(x) = -f'(x) + R.$$

$$\therefore \quad F'(x_1) = -f'(x_1) + R = 0.$$

$$R = f'(x_1).$$

$$\therefore \quad f'(x_1) = \frac{f(b) - f(a)}{b - a}.$$

§ 26·3. Another form of the above Theorem.

If we put $b - a = h$, then $a + \theta h$ is equal to a if $\theta =$ and b if $\theta = 1$.

Hence $a + \theta h$ where $0 < \theta < 1$ means an intermedi value between a and b.

So the above theorem can be written as

$$f(a + h) - f(a) = hf'(a + \theta h)$$

or $f(a + h) = f(a) + hf'(a + \theta h)$ where $0 < \theta < 1$.

Example.

If $f(x)$ is a quadratic expression, show that $\theta = \frac{1}{2}$.

Let $f(x)$ be $lx^2 + mx + n$.

Then $f'(x) = 2lx + m$.

$$f(a + h) = f(a) + hf'(a + \theta h).$$

$$\therefore \; l(a+h)^2 + m(a+h) + n = la^2 + ma + n + h\,(2\,\overline{la + \theta h} +$$

Simplifying, $2\theta lh^2 = lh^2$.

$$\therefore \quad \theta = \tfrac{1}{2}.$$

§ **26·4.** *Another Proof.* The mean value theorem can also illustrated graphically. The following is more a verification the theorem than a proof :

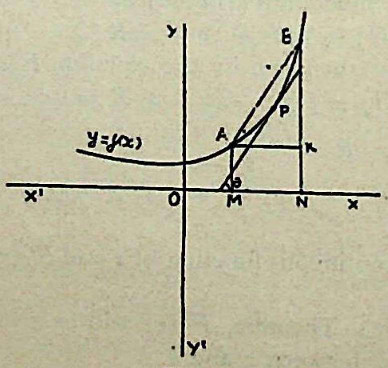

Fig. 9

Let A be the point $\{a, f(a)\}$ and B the point $\{b, f(b)\}$. The slope of the chord AB is $\dfrac{f(b)-f(a)}{b-a}$.

If $f(x)$ is continuous and has a derivative at every point between A and B, then there is at least one point P between A and B at which the tangent is parallel to AB.

Let the x coordinate of the point be x_1.

$$\therefore \quad \tan\theta = f'(x_1) = \tan\angle BAK.$$

$$\therefore \quad f'(x_1) = \frac{f(b)-f(a)}{b-a}.$$

§ 26·5. Generalised Mean Value Theorems.

If $f(x)$ and its first $(n-1)$ derivatives are continuous in $a \leqslant x \leqslant b$ and its nth derivative exists in the open interval $a < x < b$,

then $f(b) = f(a) + (b-a)f'(a) + \dfrac{(b-a)^2}{2!}f''(a) + \ldots\ldots$

$$\ldots\ldots + \frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(b-a)^n}{n!}f^n(a+\theta\,\overline{b-a})$$

for some value of θ such that $0 < \theta < 1$.

Consider the function $F(x)$ given by

$$F(x) = f(b) - f(x) - (b-x)f'(x) - \frac{(b-x)^2}{2!}f''(x) \ldots\ldots$$

$$\ldots\ldots - \frac{(b-x)^{n-1}}{(n-1)!}f^{n-1}(x) - \frac{(b-x)^n}{n!}R \qquad (1)$$

where R is given by the equation

$$F(a) = 0$$

i.e., $f(b) - f(a) - (b-a)f'(a)$

$$- \frac{(b-a)^2}{2!}f''(a) \ldots\ldots - \frac{(b-a)^n}{n!}R = 0.$$

$$F(a) = 0 \text{ and } F(b) = 0.$$

$\therefore$ $F'(x)$ vanishes for at least one value (say x_1) of x between a and b.

Now, differentiating $F(x)$, we get

$$F'(x) = -f'(x) + f'(x) - (b-x)f''(x) + (b-x)f''(x)$$

$$- \frac{(b-x)^2}{2!}f'''(x) \ldots\ldots - \frac{(b-x)^{n-1}}{(n-1)!}f^n(x) + \frac{(b-x)^{n-1}}{(n-1)!}R$$

$$= \frac{(b-x)^{n-1}}{(n-1)!}\{R - f^n(x)\}.$$

Hence the equation $F'(x_1) = 0$ becomes

$$R - f^n(x_1) = 0.$$

$$\therefore\ R = f^n(x_1).$$

Writing $x_1 = a + \theta(b - a)$ where $0 < \theta < 1$ and subs-tuting the resulting value of R in (1), we get

$$f(b) = f(a) + (b - a) f'(a) + \ldots\ldots$$

$$\ldots\ldots \frac{(b-a)^n}{n!} f^n(a + \theta\,\overline{b-a})$$

§ 26·6. Other forms.

If we put $h = b - a$ in the above result, we get

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots +$$

$$\frac{h^n}{n!} f^n(a + \theta h) \text{ where } 0 < \theta <$$

If we replace b by x in the Generalised Mean Value Theorem or Taylor's Theorem, we get

$$f(x) = f(a) + (x - a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots\ldots$$

$$\ldots\ldots + \frac{(x-a)^n}{n!} f^n(a + \theta\,\overline{x-a})$$

Putting $a = 0$ in the above result, we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^n}{n!} f^n(\theta x).$$

This result is known as Maclaurin's Theorem.

Example.

If x is positive, show that

$$x > \log(1 + x) > x - \tfrac{1}{2}x^2.$$

$$f(x) = \log(1 + x),\ f'(x) = \frac{1}{1+x}.$$

$$f(x) = f(0) + xf'(\theta x) \text{ where } 0 < \theta < 1.$$

$$f(0) = \log(1 + 0) = 0,\ f'(\theta x) = \frac{1}{1 + \theta x}.$$

$$\therefore\ \log(1 + x) = \frac{x}{1 + \theta x}.$$

$1 + \theta x$ is always greater than 1.

$$\therefore\ \log(1 + x) < x.$$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(\theta x).$$

$$f''(x) = -\frac{1}{(1+x)^2}, \; f''(\theta x) = -\frac{1}{(1+\theta x)^2}.$$

$$f'(0) = 1.$$

$$\therefore \log(1+x) = 0 + x - \frac{x^2}{2}\frac{1}{(1+\theta x)^2}.$$

$$\log(1+x) - x = -\frac{x^2}{2}\frac{1}{(1+\theta x)^2}.$$

$\dfrac{1}{(1+\theta x)^2}$ is always less than 1.

$$x - \log(1+x) = \frac{x^2}{2}\frac{1}{(1+\theta x)^2}$$

$$< \frac{x^2}{2}.$$

$$\therefore x - \frac{x^2}{2} < \log(1+x).$$

$$\therefore x > \log(1+x) > x - \frac{x^2}{2}.$$

Exercises XX.

1. Prove that $1 + x < e^x$.

2. Show that $x - \log(1+x) > \dfrac{x^2}{2(1+x)^2}$. (A.U. 44)

3. Show that $\cos x > 1 - \dfrac{x^2}{2}$.

4. Prove that $\underset{h\to 0}{\mathrm{Lt}} \dfrac{f(x+2h) - 2f(x+h) + f(x)}{h^2} = f''(x)$

provided that $f''(x)$ is continuous.

5. In the theorem $f(b) - f(a) = (b-a)f'(c)$, determine c, lying between a and b, when

(*i*) $f(x) = x^3 - 2x^2, a = 2, b = 5$. (B.A. 45 M)

(*ii*) $f(x) = x^3 + x, a = 1, b = 2$. (B.Sc. 64)

§ 27·1. Application to the theory of equations.

If $f(x)$ be a rational integral function of x, then $f(x)$ and its derivative $f'(x)$ are both continuous for all finite values of x. Hence at least one real root of the equation

$$f'(x) = 0$$

will lie between any two real roots of $f(x) = 0$.

(Rolle's Theorem)

From this we can say that at the most one real root of $f(x) = 0$ lies between any two consecutive roots of $f'(x) = 0$.

i.e., the real roots of $f'(x) = 0$ *separate those of* $f(x) = 0$.

Example.

Let us consider the function

$$f(x) = 4x^3 - 21x^2 + 18x + 20.$$

We have $f'(x) = 12x^2 - 42x + 18$
$= 6(2x-1)(x-3).$

Hence the real roots of $f'(x) = 0$ are $\frac{1}{2}$ and 3. So, the roots of $f(x) = 0$, if any, will be in the intervals between $-\infty$ and $\frac{1}{2}$; $\frac{1}{2}$ and 3; 3 and $+\infty$ respectively.

Now for $x = -\infty, \frac{1}{2}, 3, +\infty$ the signs of $f(x)$ are $-, +, -, +$ respectively, so that $f(x)$ must vanish once in each of the above intervals. Hence there are three real roots.

§ 27·2. Multiple roots. If $f(x)$ is a rational integral function of x and the equation $f(x) = 0$ has m roots equal to a, then $f(x)$ must be of the form $(x-a)^m \phi(x)$, where $\phi(a) \neq 0$.

$$\therefore\ f'(x) = (x-a)^m \phi'(x) + m(x-a)^{m-1}\phi(x)$$
$$= (x-a)^{m-1}\{(x-a)\phi'(x) + m\phi(x)\}.$$

Hence $(x-a)^{m-1}$ is a common factor of $f(x)$ and $f'(x)$ and it is easily seen that $(x-a)^{m-1}$ will not be a common factor unless $f(x)$ is divisible by $(x-a)^m$. Hence the multiple roots of $f(x)$, if any, are to be detected by finding the greatest common factors of $f(x)$ and $f'(x)$ by the usual algebraic process. We may then state a rule for finding the multiple roots of an equation $f(x) = 0$ as follows :

(1) Find $f'(x)$.
(2) Find the H.C.F. of $f(x)$ and $f'(x)$.
(3) Find the roots of the H.C.F.

Each different root of the H.C.F. will occur once more in $f(x)$ than it does in the H.C.F.

Example.

Find the multiple roots of the equation

$$x^4 - 9x^2 + 4x + 12 = 0.$$

$$f(x) = x^4 - 9x^2 + 4x + 12.$$
$$f'(x) = 4x^3 - 18x + 4.$$

The H.C.F. of $f(x)$ and $f'(x)$ is easily found to be $x - 2$; hence $(x-2)^2$ is a factor of $f(x)$. The remaining factors are easily ascertained; thus we find

$$f(x) = (x-2)^2(x+1)(x+3).$$

Exercises XXI.

1. Solve the following equations by finding their multiple roots :—

(1) $x^3 + x^2 - 16x + 20 = 0$.
(2) $x^5 - 10x^2 + 15x - 6 = 0$.
(3) $x^4 - 2x^3 - 11x^2 + 12x + 36 = 0$.
(4) $x^3 + 2x^2 - 7x + 4 = 0$.
(5) $x^4 - 6x^3 + 10x^2 - 6x + 9 = 0$.
(6) $4x^3 - 16x^2 - 19x - 5 = 0$.

2. Prove that the curves

(1) $y = x^4 - 6x^3 + 9x^2 + 4x - 12$
(2) $y = x^4 - 3x^3 + x^2 + 3x - 2$
(3) $y = 8x^3 - 44x^2 + 78x - 45$

touch the axis of x and find where they cut it.

3. If the roots of the equation $f(x) = 0$ are $a, b, c, \ldots,$ show that $\dfrac{f'(x)}{f(x)} = \dfrac{1}{x-a} + \dfrac{1}{x-b} + \dfrac{1}{x-c} + \ldots$

Hence or otherwise show that the root of $f(x) = 0$ repeated r times, is a root of $f'(x) = 0$ repeated $r - 1$ times.

Solve the equation $4x^5 + 15x^4 - 40x^2 + 48 = 0$ given that it has repeated roots.

4. If the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = 0$$

has a double root α, prove that α is a root of the equation

$$p_1 x^{n-1} + 2 p_2 x^{n-2} + 3 p_3 x^{n-3} + \ldots + n p_n = 0.$$

5. Find the limits between which the real roots of the following equations lie :—

(1) $x^3 - 6x^2 + 2 = 0$.
(2) $x^4 - 18x^2 + 12 = 0$.
(3) $x^3 - 12x^2 + 36x - 10 = 0$.
(4) $2x^3 - 3x^2 - 36x - 5 = 0$.
(5) $x^4 + 4x^3 - 20x^2 + 10 = 0$.

6. Find the nature of the roots of the equations :

(1) $3x^4 - 8x^3 - 6x^2 + 24x + 7 = 0$.
(2) $4x^3 - 21x^2 + 18x + 30 = 0$.
(3) $2x^3 - 9x^2 + 12x + 3 = 0$.

7. What is the condition for the equation $x^3 + px + q = 0$ to have two equal roots ?

§ 27·3 Newton's method of approximation.

When an approximate value of a root of an equation is know a closer approximation may be obtained by the following metho commonly ascribed to Newton. This method of approximation valuable as being applicable to numerical equations involvi transcendental functions as well as those which involve algebraic functions only.

Let α be a root of the equation $f(x) = 0$ and a i approximation to α.

$\alpha = a + h$ where h is very small.

Now $f(a + h) = f(a) + hf'(a + \theta h)$ where $0 < \theta < 1$.

$f(a + h) = 0$ since $a + h$ is a root of $f(x) = 0$.

$$\therefore \quad f(a) + hf'(a + \theta h) = 0.$$

$$\therefore \quad h = -\frac{f(a)}{f'(a + \theta h)}$$

$$= -\frac{f(a)}{f'(a)} \text{ approximately.}$$

$$\therefore \quad \alpha = a - \frac{f(a)}{f'(a)} \text{ approximately.}$$

Examples.

Ex. 1. Find an approximate value of the positive root of t equation $x^3 - 2x - 5 = 0$.

We find that the positive root lies between 2 and 3.

If $f(x) = x^3 - 2x - 5$,

$f(2) = -1, f(3) = +16$.

$\therefore$ Between 2 and 3, $f(x)$ changes its sign. Let the root o the equation be $2 + h$ where h is very small.

$$\therefore \quad h = -\frac{f(2)}{f'(2)}$$

$$f(x) = x^3 - 2x - 5\,;\; f'(x) = 3x^2 - 2.$$

$$f(2) = -1\,;\; f'(2) = 10.$$

$$\therefore \quad h = \frac{1}{10} = \cdot 1.$$

So the first approximation is $2{\cdot}1$.

If $2{\cdot}1 + h$ is the root of the equation, then

$$h = -\frac{f(2{\cdot}1)}{f'(2{\cdot}1)} \text{ approximately}$$

$$= \frac{-\cdot 061}{11{\cdot}23} = -\cdot 00543.$$

So the second approximation is $2{\cdot}1 - \cdot 00543 = 2{\cdot}0946$.

Ex. 2. Show by starting from the rough approximation $x = \pi$ as a root of the equation $\sin x = ax$ where a is small, that a much better approximation can be obtained as $x = (1 - a + a^2)\pi$.

$$f(x) = \sin x - ax.$$
$$f'(x) = \cos x - a.$$
$$\frac{f(x)}{f'(x)} = \frac{\sin x - ax}{\cos x - a}.$$
$$h = -\frac{f(\pi)}{f'(\pi)} = -\frac{a\pi}{1+a}$$
$$= -a\pi(1+a)^{-1}$$
$$= -a\pi(1 - a + \ldots)$$
$$= -a\pi + a^2\pi \text{ approximately.}$$

The approximate value of the root is $\pi - a\pi + a^2\pi$

i.e., $\pi(1 - a + a^2)$.

Exercises XXII.

1. Find the root of $x^3 - 4x^2 + 7x + 24 = 0$ which lies between 2 and 3.

2. One root of the equation $x^4 - 12x^2 - 12x - 3 = 0$ is approximately equal to 4. Find it correct to two places of decimals.

3. Find the positive root of the equation $x^3 - 8x - 40 = 0$ correct to two places of decimals.

4. It is known that a root of the equation $x^3(1 + x) = 5$ does not differ much from 1·3. Find an approximation correct to three decimal places.

5. Show that the equation $x^3 + 3x - 7 = 0$ has only one real root and determine it to two places of decimals.

6. A cubical cistern contains 800 c. ft. of water so that its inside edge is between 9 and 10 ft. long. Find the length of the inside edge approximately.

7. If $\sin(30° + \theta) = {\cdot}51$, prove, without using tables, that $\theta = 40'$ approximately.

§ 28·1. Indeterminate forms.

The limit of $\dfrac{f(x)}{F(x)}$ as $x \to a$ is, *in general*, equal to the limit of numerator divided by the limit of the denominator. But when these limits are both zero or both infinity, the quotient reduces

to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ which is indeterminate. In the present section, we shall consider the method of evaluating the limits in such and other similar cases. Several cases are to be considered *viz.*, when upon substitution of the assigned value of the independent variable, the function reduces to one of the forms :

$$\frac{0}{0},\quad 0\times\infty,\quad \frac{\infty}{\infty},\quad \infty-\infty,\quad 0^0,\quad \infty^0,\quad 1^\infty.$$

§ 28·2. Evaluation of the indeterminate form $\frac{0}{0}$.

Hospital's Rule.

If a function of the form $\frac{f(x)}{F(x)}$ be such that $f(a)=0$ and $F(a)=0$, the function takes the indeterminate form $\frac{0}{0}$ when a is substituted for x. It is then required to find $\underset{x\to a}{\text{Lt}}\frac{f(x)}{F(x)}$.

Let $x=a+h$; as $x\to a$, $h\to 0$.

$$\therefore \underset{x\to 0}{\text{Lt}}\frac{f(x)}{F(x)}=\underset{h\to 0}{\text{Lt}}\frac{f(a+h)}{F(a+h)}$$

$$=\underset{h\to 0}{\text{Lt}}\frac{f(a)+hf'(a+\theta_1 h)}{F(a)+hF'(a+\theta_2 h)}.$$

where $0<\theta_1<1$, $0<\theta_2<1$.

Since $f(a)=0$, and $F(a)=0$.

$$\underset{x\to a}{\text{Lt}}\frac{f(x)}{F(x)}=\underset{h\to 0}{\text{Lt}}\frac{f'(a+\theta_1 h)}{F'(a+\theta_2 h)}$$

$$=\frac{f'(a)}{F'(a)}=\underset{x\to a}{\text{Lt}}\frac{f'(x)}{F'(x)}.$$

Here if $f'(a)=0$, $F'(a)=0$.

$$\underset{x\to a}{\text{Lt}}\frac{f'(x)}{F'(x)}=\underset{x\to a}{\text{Lt}}\frac{f''(x)}{F''(x)}=\frac{f''(a)}{F''(a)}.$$

Here also if $f''(a)=F''(a)=0$.

$$\underset{x\to a}{\text{Lt}}\frac{f(x)}{F(x)}=\underset{x\to a}{\text{Lt}}\frac{f'''(x)}{F'''(x)}.$$

We can continue this process, till we get rid of the indeterminate form.

Examples.

Ex. 1. Find the limit as x tends to zero of

$$\frac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}.$$

$$\underset{x\to 0}{\text{Lt}} \frac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x} \left(\text{is of the form } \frac{0}{0}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{2\cos 2x + 4\sin x\cos x - 2\cos x}{-\sin x + 2\sin x\cos x} \left(\text{is of the form } \frac{0}{0}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{-4\sin 2x + 4\cos^2 x - 4\sin^2 x + 2\sin x}{-\cos x - 2\sin^2 x + 2\cos^2 x}$$

$$= \frac{0 + 4 - 0 + 0}{-1 - 0 + 2} = \frac{4}{1} = 4.$$

Ex. 2. Evaluate $\underset{x\to 0}{\text{Lt}} \dfrac{\sin\log(1+x)}{\log(1+\sin x)}$.

$$\underset{x\to 0}{\text{Lt}} \frac{\sin\log(1+x)}{\log(1+\sin x)} \left(\text{is of the form } \frac{0}{0}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\dfrac{\cos\log(1+x)}{1+x}}{\dfrac{\cos x}{1+\sin x}} = \frac{\dfrac{\cos\log 1}{1}}{\dfrac{\cos 0}{1+0}} = 1.$$

Exercises XXIII.

Evaluate the following limits :—

1. $\underset{x\to 1}{\text{Lt}} \dfrac{1-x}{\log x}$.

2. $\underset{x\to 4}{\text{Lt}} \dfrac{x^2 - 16}{x^2 + x - 20}$.

3. $\underset{x\to 0}{\text{Lt}} \dfrac{e^2 - 1 + \log(1-x)}{\sin^3 x}$.

4. $\underset{x\to 0}{\text{Lt}} \dfrac{e^{3x} + e^{-3x} - 2}{5x^2}$.

5. $\underset{x\to 0}{\text{Lt}} \dfrac{\sqrt{2+x} - \sqrt{2}}{x}$.

6. $\underset{x\to 0}{\text{Lt}} \dfrac{e^x + \sin x - 1}{\log(1+x)}$.

7. $\underset{x\to 0}{\text{Lt}} \dfrac{\sin x - \log(1+x) - x^2 e^x}{x^3}$.

8. $\underset{x\to 0}{\text{Lt}} \dfrac{\tan x - \sin x}{\sin^3 x}$.

9. $\underset{x\to \pi/4}{\text{Lt}} \dfrac{\sec^2 x - 2\tan x}{1 + \cos 4x}$.

10. $\underset{x\to 1}{\text{Lt}} \dfrac{\sqrt{x} - 1 + \sqrt{x-1}}{\sqrt{x^2 - 1}}$. (B.Sc. 52 M)

11. $\underset{x\to 0}{\text{Lt}} \dfrac{x\cos x - \log(1+x)}{x^2}$. (B.Sc. 52 M)

12. $\underset{x \to 0}{\text{Lt}} \dfrac{a^x - b^x}{x}$. (B.Sc. 52 M)

13. $\underset{x \to a}{\text{Lt}} \dfrac{\log(\sin x \operatorname{cosec} a)}{\log(\cos a \sec x)}$. (B.A. 49 M)

14. $\underset{x \to 1}{\text{Lt}} \dfrac{1 - x + \log x}{1 - (2x - x^2)^{1/2}}$.

15. $\underset{x \to 1}{\text{Lt}} \dfrac{x^x - x}{1 - x + \log x}$.

16. $\underset{x \to 0}{\text{Lt}} \dfrac{\log(1 + kx^2)}{1 - \cos x}$.

17. $\underset{x \to 0}{\text{Lt}} \dfrac{\sin 2x + a \sin x}{x^3}$ is finite, find the value of and the limit.

18. $\underset{x \to \pi}{\text{Lt}} \dfrac{x \cos x + \pi}{\sin x}$.

19. $\underset{x \to 0}{\text{Lt}} \dfrac{2x - 3 \sin x + x \cos}{x^3 (1 - \cos 2x)}$

§ 28·3. Indeterminate form $\dfrac{\infty}{\infty}$.

Let $\underset{x \to a}{\text{Lt}} f(x)$ be ∞ and $\underset{x \to a}{\text{Lt}} F(x)$ be ∞.

$$\underset{x \to a}{\text{Lt}} \frac{f(x)}{F(x)} = \underset{x \to a}{\text{Lt}} \frac{\dfrac{1}{F(x)}}{\dfrac{1}{f(x)}} \qquad \left(\text{form } \frac{0}{0}\right.$$

$$= \underset{x \to a}{\text{Lt}} \frac{-\dfrac{F'(x)}{\{F(x)\}^2}}{\dfrac{-f'(x)}{\{f(x)\}^2}}$$

$$= \underset{x \to a}{\text{Lt}} \frac{F'(x)}{f'(x)} \cdot \frac{\{f(x)\}^2}{\{F(x)\}^2}$$

$$= \underset{x \to a}{\text{Lt}} \frac{F'(x)}{f'(x)} \cdot \left\{ \underset{x \to a}{\text{Lt}} \frac{f(x)}{F(x)} \right\}^2.$$

Let us assume that $\underset{x \to a}{\text{Lt}} \dfrac{f(x)}{F(x)} = k$.

Case 1. $k \neq 0, x \neq \infty$.

$$\therefore \quad k = \underset{x \to a}{\text{Lt}} \frac{F'(x)}{f'(x)} k^2.$$

$$\therefore \quad \frac{1}{k} = \underset{x \to a}{\text{Lt}} \frac{F'(x)}{f'(x)}.$$

$$\therefore \quad \underset{x \to a}{\text{Lt}} \frac{f(x)}{F(x)} = k = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{F'(x)}.$$

Case 2. $k = 0$.

$$k = \operatorname*{Lt}_{x \to a} \frac{f(x)}{F(x)}.$$

$$k + 1 = \operatorname*{Lt}_{x \to a} \frac{f(x)}{F(x)} + 1 = \operatorname*{Lt}_{x \to a} \frac{f(x) + F(x)}{F(x)}$$

$$= \operatorname*{Lt}_{x \to a} \frac{f'(x) + F'(x)}{F'(x)} \text{ (by Case 1)}$$

$$= \operatorname*{Lt}_{x \to a} \frac{f'(x)}{F'(x)} + 1$$

$$\therefore \quad k = \operatorname*{Lt}_{x \to a} \frac{f'(x)}{F'(x)}.$$

Case 3. $k = \infty$.

$$\text{Here } \frac{1}{\operatorname*{Lt}_{x \to a} \frac{f(x)}{F(x)}} = \operatorname*{Lt}_{x \to a} \frac{F(x)}{f(x)}$$

$$= \operatorname*{Lt}_{x \to a} \frac{F'(x)}{f'(x)} \text{ (by Case 2)}$$

$$\therefore \quad \operatorname*{Lt}_{x \to a} \frac{f(x)}{F(x)} = \operatorname*{Lt}_{x \to a} \frac{f'(x)}{F'(x)}.$$

Hence, in all cases, $\operatorname*{Lt}_{x \to a} \frac{f(x)}{F(x)} = \operatorname*{Lt}_{x \to a} \frac{f'(x)}{F'(x)}$.

In most cases coming under the form $\frac{\infty}{\infty}$ it is necessary to convert into the form $\frac{0}{0}$ as soon as possible ; otherwise the process of differentiating the numerator and denominator will never terminate. Thus if we once have $\frac{1}{x}$ in the numerator or in the denominator and the limit when $x \to 0$ has to be found, successive differentiations would involve $\frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^4}, \ldots\ldots$ which will tend to infinity as $x \to 0$. So a transformation to the form $\frac{0}{0}$ has to be made at a convenient stage.

Examples.

Ex. 1. Evaluate $\underset{x\to 0}{\text{Lt}} \dfrac{\log x}{\operatorname{cosec} x}$.

$$\underset{x\to 0}{\text{Lt}} \frac{\log x}{\operatorname{cosec} x} \left(\text{is of the form } \frac{\infty}{\infty}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{-\sin^2 x}{x \cos x} \left(\text{is of the form } \frac{0}{0}\right)$$

$$= \underset{x\to 0}{\text{Lt}} -\frac{2 \sin x \cos x}{\cos x - x \sin x}$$

$$= -\frac{0}{1} = 0.$$

Ex. 2. Find $\underset{x\to \infty}{\text{Lt}} \dfrac{x^2}{e^{x^2}}$.

$$= \underset{x\to \infty}{\text{Lt}} \frac{x^2}{e^{x^2}} \left(\text{is of the form } \frac{\infty}{\infty}\right)$$

$$= \underset{x\to \infty}{\text{Lt}} \frac{2x}{2xe^{x^2}} = \underset{x\to \infty}{\text{Lt}} \frac{1}{e^{x^2}} = 0.$$

Exercises XXIV.

Evaluate the following limits :—

1. $\underset{x\to \infty}{\text{Lt}} \dfrac{5x-2}{3x+4}$.
2. $\underset{x\to \infty}{\text{Lt}} \dfrac{3x^2-2x+1}{2x^2+5x-2}$.
3. $\underset{x\to 0}{\text{Lt}} \dfrac{\log x}{\cot x}$.
4. $\underset{x\to 0}{\text{Lt}} \dfrac{\cot 2x}{\cot x}$.
5. $\underset{x\to 1}{\text{Lt}} \dfrac{\log (1-x)}{\cot \pi x}$.
6. $\underset{x\to \frac{1}{2}}{\text{Lt}} \dfrac{\sec \pi x}{\tan 3\pi x}$.
7. $\underset{x\to 0}{\text{Lt}} \dfrac{\log \sin 2x}{\log \sin x}$.
8. $\underset{x\to \frac{\pi}{2}}{\text{Lt}} \dfrac{\log \left(x-\frac{\pi}{2}\right)}{\tan x}$.
9. $\underset{x\to \infty}{\text{Lt}} \dfrac{x \cos \frac{1}{x}}{1+x}$.
10. $\underset{x\to \infty}{\text{Lt}} \dfrac{e^x}{x^3}$.
11. $\underset{x\to 0}{\text{Lt}} \dfrac{\log x^2}{\cot x^2}$.

§ 28·4. Indeterminate form $0 \times \infty$.

Let $\underset{x \to a}{\text{Lt}} f(x)$ be 0 and $\underset{x \to a}{\text{Lt}} F(x)$ be ∞. $f(x) F(x)$ can be written as $\dfrac{f(x)}{\dfrac{1}{F(x)}}$ or $\dfrac{F(x)}{\dfrac{1}{f(x)}}$.

$\therefore$ $\underset{x \to a}{\text{Lt}} f(x) F(x)$ is reduced to the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

Examples.

Ex. 1. Evaluate $\underset{x \to \pi/2}{\text{Lt}} \sec 3x \cos 5x$.

$$\underset{x \to \pi/2}{\text{Lt}} \sec 3x \cos 5x \text{ (is of the form } \infty \times 0)$$

$$= \underset{x \to \pi/2}{\text{Lt}} \frac{\cos 5x}{\cos 3x} \left(\text{ ,, } \quad \text{ ,, } \quad \frac{0}{0} \right)$$

$$= \underset{x \to \pi/2}{\text{Lt}} \frac{-5 \sin 5x}{-3 \sin 3x} = -\frac{5}{3}.$$

Ex. 2. Find $\underset{x \to 0}{\text{Lt}} \tan x \log x$.

$$\underset{x \to 0}{\text{Lt}} \tan x \log x \qquad \text{(form } 0 \times \infty)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\log x}{\cot x} \qquad \left(\text{ ,, } \quad \frac{\infty}{\infty} \right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{x}}{-\operatorname{cosec}^2 x} \qquad \left(\text{ ,, } \quad \frac{\infty}{\infty} \right)$$

$$= \underset{x \to 0}{\text{Lt}} -\frac{\sin^2 x}{x} \qquad \left(\text{ ,, } \quad \frac{0}{0} \right)$$

$$= \underset{x \to 0}{\text{Lt}} -\frac{2 \sin x \cos x}{1} = 0.$$

§ 28·5. Indeterminate form $\infty - \infty$.

It is possible in general to transform the expression into a fraction which will assume either the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

Example.

Evaluate $\underset{x\to 0}{\text{Lt}}\left(\frac{1}{x}-\cot x\right)$.

$$\underset{x\to 0}{\text{Lt}}\left(\frac{1}{x}-\cot x\right) \text{ (is of the form } \infty - \infty)$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\sin x - x\cos x}{x\sin x}\left(\text{is of the form } \frac{0}{0}\right)$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\cos x - \cos x + x\sin x}{x\cos x + \sin x}$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{x\sin x}{x\cos x + \sin x}\left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{x\cos x + \sin x}{\cos x - x\sin x + \cos x}$$

$$= \frac{0}{2} = 0.$$

Exercises XXV.

Evaluate the following :—

1. $\underset{x\to n}{\text{Lt}}\ (x-n)\cot \pi x$.
2. $\underset{x\to n}{\text{Lt}}\ (x-n)\operatorname{cosec} n\pi$.
3. $\underset{x\to a}{\text{Lt}}\ \log\left(2-\frac{x}{a}\right)\cot(x-a)$.
4. $\underset{x\to \infty}{\text{Lt}}\ x\tan\frac{1}{x}$.
5. $\underset{x\to 0}{\text{Lt}}\ x\log x$.
6. $\underset{x\to 1}{\text{Lt}}\ \tan^2\left(\frac{\pi x^2}{2}\right)(1+\sec \pi x)$.
7. $\underset{x\to \infty}{\text{Lt}}\ x^n e^{-x}\ (n>0)$. (B.Sc. 51 M)
8. $\underset{x\to 0}{\text{Lt}}\ x^m(\log x)^n$. (B.Sc. 51 M)
9. $\underset{\theta\to \pi}{\text{Lt}}\ \log(\theta-\pi)\tan\theta$. (B.Sc. 50 M)
10. $\underset{x\to \pi/2}{\text{Lt}}\ (\sec x - \tan x)$.
11. $\underset{x\to 0}{\text{Lt}}\left(\frac{1}{x^2}-\cot^2 x\right)$.
12. $\underset{x\to \pi/2}{\text{Lt}}\left(x\tan x - \frac{\pi}{2}\sec x\right)$.
13. $\underset{x\to 0}{\text{Lt}}\left\{\frac{1}{x}-\frac{\log(1+x)}{x^2}\right\}$. (B.Sc. 50 M)
14. $\underset{x\to \pi/2}{\text{Lt}}\left\{\sec x - \frac{1}{1-\sin x}\right\}$.
15. $\underset{x\to \pi/2}{\text{Lt}}\left(\tan x - \frac{2x\sec x}{\pi}\right)$.

§ 28·6. Indeterminate forms. $0^0, 1^\infty, \infty^0, 0^\infty$.

If $K = \underset{n\to a}{\text{Lt}}\ f(x)^{F(x)}$ takes one of these forms, then taking logarithm on both sides, we get $\log K = \underset{x\to a}{\text{Lt}}\ F(x)\log f(x)$.

Then $\log K$ will take on the indeterminate form $0 \times \infty$. We can evaluate this by § 28·4. This being equal to the logarithm of the limit of the function, the limit of the function is known.

Examples.

Ex. 1. Evaluate $\underset{x\to\pi/4}{\text{Lt}} (\tan x)^{\tan 2x}$.

$$K = \underset{x\to\pi/4}{\text{Lt}} (\tan x)^{\tan 2x}. \qquad \text{(form } 1^\infty)$$

$$\log K = \underset{x\to\pi/4}{\text{Lt}} \tan 2x \log(\tan x) \qquad \text{(form } \infty \times 0)$$

$$= \underset{x\to\pi/4}{\text{Lt}} \frac{\log(\tan x)}{\cot 2x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x\to\pi/4}{\text{Lt}} \frac{\dfrac{\sec^2 x}{\tan x}}{-2\operatorname{cosec}^2 2x}$$

$$= \underset{x\to\pi/4}{\text{Lt}} \frac{-\sin^2 2x}{2\tan x \cos^2 x}$$

$$= \underset{x\to\pi/4}{\text{Lt}} -2\sin x\cos x$$

$$= -1.$$

$$K = e^{-1} = \frac{1}{e}.$$

Ex. 2. Evaluate $\underset{x\to 1}{\text{Lt}} (1-x^2)^{\frac{1}{\log(1-x)}}$.

$$K = \underset{x\to 1}{\text{Lt}} (1-x^2)^{\frac{1}{\log(1-x)}}. \qquad \text{(form } 0^0)$$

$$\log K = \underset{x\to 1}{\text{Lt}} \frac{\log(1-x^2)}{\log(1-x)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x\to 1}{\text{Lt}} \frac{-\dfrac{2x}{1-x^2}}{-\dfrac{1}{1-x}}$$

$$= \underset{x\to 1}{\text{Lt}} \frac{2x}{1+x} = 1.$$

$$K = e.$$

Ex. 3. Evaluate $\underset{x\to\infty}{\text{Lt}} (1+x^2)^{e^{-x}}$.

$$K = \underset{x\to\infty}{\text{Lt}} (1+x^2)^{e^{-x}}. \qquad \text{(form } \infty^0)$$

$$\log K = \underset{x\to\infty}{\text{Lt}}\, e^{-x} \log (1 + x^2) \qquad \text{(form } 0 \times$$

$$= \underset{x\to\infty}{\text{Lt}}\, \frac{\log (1 + x^2)}{e^x} \qquad \left(\text{form } \frac{\infty}{\infty}\right.$$

$$= \underset{x\to\infty}{\text{Lt}}\, \frac{2x}{(1 + x^2)\cdot e^x} \qquad \left(\text{form } \frac{\infty}{\infty}\right.$$

$$= \underset{x\to\infty}{\text{Lt}}\, \frac{2}{(1 + x^2)\, e^x + e^x\, 2x}$$

$$= 0.$$

$$K = e^0 = 1.$$

Ex. 4. $\underset{x\to 0}{\text{Lt}} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$. (B.A. 52

$K = \underset{x\to 0}{\text{Lt}} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$. Since $\frac{\tan x}{x} \to 1$ as $x \to 0$, it is the form 1^∞.

$$\log K = \underset{x\to 0}{\text{Lt}}\, \frac{1}{x^2} \log \left(\frac{\tan x}{x}\right) \qquad \left(\text{form}\right.$$

$$= \underset{x\to 0}{\text{Lt}} \left\{\frac{x}{\tan x} \cdot \frac{x \sec^2 x - \tan x}{x^2} \Big/ 2x\right\}$$

$$= \underset{x\to 0}{\text{Lt}}\, \frac{x \sec^2 x - \tan x}{2x^2 \tan x} \qquad \left(\text{form}\right.$$

$$= \underset{x\to 0}{\text{Lt}}\, \frac{\sec x^2 + 2x \sec^2 x \tan x - \sec^2 x}{2x^2 \sec^2 x + 4x \tan x}$$

$$= \underset{x\to 0}{\text{Lt}}\, \frac{\sec^2 x \tan x}{x \sec^2 x + 2 \tan x}$$

$$= \underset{x\to 0}{\text{Lt}}\, \frac{\sec^4 x + 2 \sec^2 x \tan^2 x}{2x \sec^2 x \tan x + 3 \sec^2 x} = \frac{1}{3}.$$

$$\therefore\ K = e^{1/3}.$$

Exercises XXVI.

Evaluate the following limits :—

1. $\underset{x\to 0}{\text{Lt}}\, x^x$.
2. $\underset{x\to 1}{\text{Lt}}\, (2 - x)^{\tan \frac{\pi x}{2}}$.
3. $\underset{x\to 0}{\text{Lt}}\, (\cos x)^{\cot^2 x}$.
4. $\underset{x\to 0}{\text{Lt}} \left(\frac{x}{e^x - 1}\right)^{\frac{1}{x}}$.

5. $\underset{x\to 0}{\text{Lt}} \left(\frac{1+x}{1-x}\right)^{\frac{1}{x}}$.

6. $\underset{x\to 1}{\text{Lt}}\ x^{\frac{1}{1-x}}$.

7. $\underset{x\to 0}{\text{Lt}} (\cos x)^{\operatorname{cosec}^2 x}$.

8. $\underset{x\to a}{\text{Lt}} \left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$.

9. $\underset{x\to \pi/2}{\text{Lt}} (\sin x)^{2\tan x}$.

10. $\underset{x\to 0}{\text{Lt}} \left(\log\frac{1}{x}\right)^{\log(1-x)}$.

11. $\underset{x\to 0}{\text{Lt}} (\cos \alpha x)^{\frac{1}{x}}$.

12. $\underset{x\to 0}{\text{Lt}} (\cos \alpha x)^{\frac{1}{x^2}}$.

13. $\underset{x\to 0}{\text{Lt}} (\cot x)^{\sin x}$.

(B.A. 52 M)

14. $\underset{x\to 0}{\text{Lt}} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$.

(B.Sc. 52 M)

15. $\underset{x\to \pi/2}{\text{Lt}} (\sin x)^{\sec^2 x}$.

(B.A. 45 M)

16. $\underset{x\to 0}{\text{Lt}} (\sin x)^{\tan x}$.

(B.A. 51 M)

17. $\underset{x\to 0}{\text{Lt}} \left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$.

CHAPTER VI

MAXIMA AND MINIMA

§ 29·1. We shall discuss in this chapter the maximum a minimum values of a function when the function and its derivati are continuous. We shall first define the maximum and minim values of a function.

If a continuous function increases up to a value and t decreases, that value is called a *maximum value* of the functi Similarly, if a continuous function decreases to a value and t increases, that value is called a *minimum value* of the functi We say that the value of $f(a)$ assumed by $f(x)$ when $x=$ a maximum if $f(a)$ is greater than any other value assumed by f in the immediate neighbourhood of $x = a$, *i.e.*, if we can find interval $(a-h, a+h)$ of values of x such that $f(a) > f(x)$ w $a - h < x < a$ and when $a < x < a + h$ where h is an arbitr positive number. Similarly we define a minimum ; if in interval $(a-h, a+h)$, $f(a) < f(x)$, $f(a)$ is said to be a minim value of $f(x)$. Thus in the figure the points P correspond maxima, the points Q to minima of the function whose grap shown below :

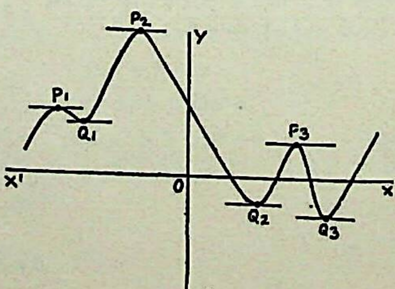

Fig. 10

It is to be noticed that (1) a maximum value is not nec sarily the greatest value the function can have ; nor a minim the least ; (2) the maxima and minima values occur alternat

§ 29·2. **Theorem 1.** A necessary condition for a maximum or a minimum value of $f(x)$ at $x = a$ is that $f'(a) = 0$.

If $f(a)$ is a maximum value of $f(x)$, then as x increases from $a - h$ to a, $f(x)$ is increasing and therefore $f'(x)$ is positive. On the other hand, as x increases from a to $a + h$, $f(x)$ is decreasing and therefore $f'(x)$ is negative. Hence as x increases through a, $f'(x)$ must change from a positive to a negative value. Conversely, if as x increases through a, $f'(x)$ changes from a positive to a negative value, $f(a)$ will be a maximum value of $f(x)$.

Hence $f(a)$ will be a maximum value of $f(x)$ if and only if $f'(x)$ changes from a positive to a negative value as x increases through a.

In the same way, it will be seen that if $f(a)$ will be a minimum value of $f(x)$ if and only if $f'(x)$ changes from a negative to a positive value as x increases through a.

$f'(x)$ is continuous and a continuous function can change sign only by passing through the value zero. Therefore, if $f(a)$ is a turning value of $f(x)$, $f'(a)$ will be zero.

§ 29·3. **Theorem 2.** If $f'(a) = 0$ and $f''(a) \neq 0$, then $f(x)$ has a maximum if $f''(a) < 0$ and a minimum if $f''(a) > 0$.

If $f(a)$ is a maximum value of $f(x)$, $f'(x)$ changes from a positive to a negative value as x increases through a.

Consider $f'(x)$ as a function of x. In the interval $(a-h, a+h)$, $f'(x)$ decreases continuously changing from a positive to a negative value. Hence its derivative $f''(x)$ is negative. Therefore at the maximum point $x = a$, $f''(a)$ is negative. Similarly when $f(x)$ attains a minimum value at $x = a$, $f''(a)$ is positive. For $f'(x)$ increases continuously in the interval $(a - h, a + h)$ changing from a negative to a positive value and hence its derivative $f''(x)$ is positive.

Rule for determining the maxima and minima values of $f(x)$ when $f(x)$ and $f'(x)$ are continuous.

The roots of the equation $f'(x) = 0$ are, in general, the values of x which make $f(x)$ a maximum or a minimum. Let a be a root of $f'(x) = 0$; then $f(a)$ will be a maximum value of $f(x)$ if $f''(a)$ is negative and a minimum if $f''(a)$ is positive.

§ 29·4. A geometrical proof for the above tw theorems.

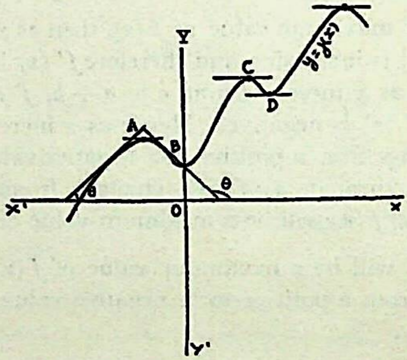

Fig. 11

In this figure the ordinates of A and C represent the maxim values and the ordinates of B and D represent the minim values of $f(x)$.

If the tangent at (x, y) on the curve make an angle θ wi the positive direction of the x-axis, we have seen that

$$\tan\theta = \frac{dy}{dx}.$$

The tangents at A, B, C, D are parallel to the x-axis.

Therefore $\theta = 0$.

$$\therefore \text{ At } A, B, C, D, \ \frac{dy}{dx} = 0.$$

For the points just to the left of A and C on the curve, θ an acute angle.

$$\therefore \ \tan\theta \text{ is } +\text{ve}, \quad i.e., \quad \frac{dy}{dx} \text{ is } +\text{ve}.$$

For the points just to the right of A and C on the curve, θ an obtuse angle.

$$\therefore \ \tan\theta \text{ is } -\text{ve}, \quad i.e., \quad \frac{dy}{dx} \text{ is } -\text{ve}.$$

Therefore in passing through a maximum value, $\frac{dy}{dx}$ chang from positive to negative. In a similar manner, we can sh that in passing through a minimum, $\frac{dy}{dx}$ changes from negati

to positive. So $\frac{d^2y}{dx^2}$ is negative at a maximum value and positive at a minimum value.

§ 29·5. The above conclusions, when $f(x)$ and its derivatives are continuous at a may be deduced from the theorem of mean value.

Proof :

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots$$

$$\ldots + \frac{h^n}{n!}f^n(a+\theta h) \text{ where } 0 < \theta < 1.$$

$$\therefore\ f(a+h) - f(a) = h\left[f'(a) + \frac{h}{2!}f''(a) + \ldots\right.$$

$$\left.\ldots \frac{h^{n-1}}{n!}f^n(a+\theta h)\right]. \quad (1)$$

If $f(x)$ is a maximum at $x = a$, then, by definition, $f(a+h) - f(a)$ and therefore the right hand side of the equation must be negative for sufficiently small values of h whether h is positive or negative.

$$f'(a) + \frac{h}{2!}f''(a) + \ldots + \frac{h^{n-1}}{n!}f^n(a+\theta h)$$

is a finite expression which does not tend to infinity as h tends to zero.

If $f'(a) \neq 0$, $f(a+h) - f(a)$ has one sign when h is positive and another when h is negative. The second member of (1) is of invariable sign for a maximum or a minimum since for a maximum $f(a+h) - f(a)$ is negative and for a minimum $f(a+h) - f(a)$ is positive.

$$\therefore\ f'(a) = 0.$$

If $f'(a) = 0$, then

$$f(a+h) - f(a) = h^2\left[\frac{f''(a)}{2!} + \frac{h}{3!}f'''(a) + \ldots\right.$$

$$\left.\ldots \frac{h^{n-2}}{n!}f^n(a+\theta h)\right].$$

The sign of the expression within the brackets on the right hand side is governed by $f''(a)$.

Since h^2 is always positive, $f(a+h) - f(a)$ is negative if $f''(a)$ is negative and $f(a+h) - f(a)$ is positive if $f''(a)$ is positive.

$\therefore$ If $f''(a) < 0$, $f(x)$ has a maximum at $x = a$.
If $f''(a) > 0$, $f(x)$ has a minimum at $x = a$.

Here we assume that $f(x), f'(x), \ldots, f^n(x)$ are continuous. In such cases as we are likely to meet with at present, the condition is generally satisfied.

Examples.

Ex. 1. Find the maxima and minima of the function

$$2x^3 - 3x^2 - 36x + 10.$$

Let $f(x)$ be $2x^3 - 3x^2 - 36x + 10$.

At the maximum or minimum, $f'(x) = 0$.

$$f'(x) = 6x^2 - 6x - 36$$
$$= 6(x - 3)(x + 2).$$

$x = 3$ and $x = -2$ give maximum or minimum.

To distinguish between the maximum and the minimum,

$$f''(x) = 6(2x - 1).$$

When $x = 3$, $f''(x) = 6(6 - 1) = 30$, *i.e.*, +ve.

When $x = -2$, $f''(x) = 6(-4 - 1) = -30$, *i.e.*, –ve.

$\therefore$ $x = -2$ gives the maximum and $x = +3$ gives the minimum.

Maximum value $= f(-2) = 54$.

Minimum value $= f(3) = -71$.

Ex. 2. Find the maximum value of $\dfrac{\log x}{x}$ for positive values of x.

Let $f(x)$ be $\dfrac{\log x}{x}$.

$$f'(x) = \frac{1 - \log x}{x^2}.$$

$$f''(x) = \frac{-3 + 2\log x}{x^3}.$$

At a maximum or a minimum, $f'(x) = 0$.

$\therefore$ $1 - \log x = 0$. $\therefore$ $x = e$.

$$f''(e) = \frac{-3 + 2\log e}{e^3} = -\frac{1}{e^3}, \quad i.e., \quad -\text{ve}.$$

$\therefore$ $x = e$ gives a maximum.

Maximum value of the function $= f(e) = \dfrac{1}{e}$.

Ex. 3. Show that the least value of $a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x$ is $(a + b)^2$. (B.Sc. 52 T.U.)

Let $f(x)$ be $a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x$.

$$f'(x) = 2a^2 \sec^2 x \tan x - 2b^2 \operatorname{cosec}^2 x \cot x$$

$$= 2\frac{a^2 \sin^4 x - b^2 \cos^4 x}{\cos^3 x \sin^3 x}.$$

$$f''(x) = 2\frac{4 \sin^4 x \cos^4 x (a^2 \sin^2 x + b^2 \cos^2 x)}{\cos^6 x \sin^6 x}$$

$$- \frac{(a^2 \sin^4 x - b^2 \cos^4 x)\frac{d}{dx}(\sin^3 x \cos^3 x)}{\cos^6 x \sin^6 x}.$$

At the maximum or minimum, $f'(x) = 0$.

$$\therefore \quad a^2 \sin^4 x - b^2 \cos^4 x = 0.$$

Then $f''(x) = \dfrac{8 \sin^4 x \cos^4 x (a^2 \sin^2 x + b^2 \cos^2 x)}{\cos^6 x \sin^6 x}$

$$= 8(a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x)$$

$$= +\text{ve expression.}$$

$\therefore \quad a^2 \sin^4 x - b^2 \cos^4 x = 0$ gives a minimum.

$$\therefore \quad \tan^2 x = \frac{b}{a}.$$

The least value of $f(x)$ is given when $\tan^2 x = \dfrac{b}{a}$.

$$\therefore \quad f(x) = a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x$$

$$= a^2(1 + \tan^2 x) + b^2(1 + \cot^2 x)$$

$$= a^2\left(1 + \frac{b}{a}\right) + b^2\left(1 + \frac{a}{b}\right)$$

$$= (a + b)^2.$$

Ex. 4. The greatest value of $ax + by$ where x and y are positive and $x^2 + xy + y^2 = 3k^2$ is $2k\sqrt{a^2 - ab + b^2}$.

Let $u = ax + by$.

u attains a maximum or a minimum when

$$\frac{du}{dx} = 0 \text{ and } \frac{d^2u}{dx^2} \text{ is } -\text{ve or } +\text{ve}.$$

$$a + b\frac{dy}{dx} = 0. \qquad (1)$$

$$x^2 + xy + y^2 = 3k^2.$$

Differentiating the above equation, we get

$$(2x + y) + (x + 2y)\frac{dy}{dx} = 0. \qquad (2)$$

Equating the two values of $\frac{dy}{dx}$, we ge

$$-\frac{a}{b}=-\frac{2x+y}{x+2y}.$$

Solving for y, $\quad y=\frac{a-2b}{b-2a}x.$ (3)

Differentiating equation (2) once again, we get

$$2+2\frac{dy}{dx}+2\left(\frac{dy}{dx}\right)^2+(x+2y)\frac{d^2y}{dx^2}=0.$$

Substituting the values of $\frac{dy}{dx}$ and y from (1) and (3), we get

$$\frac{d^2y}{dx^2}=\frac{2}{3}\frac{a^2-ab+b^2}{b^2}\frac{b-2a}{x}.$$

$\frac{d^2y}{dx^2}$ is negative for a maximum.

$\frac{b-2a}{x}$ is $-$ve since $\frac{a^3-ab+b^2}{b^3}$ is $+$ve.

$$x^2+y^2+xy=3k^2.$$

Substituting the value for y from (3), we get

$$x\sqrt{a^2-ab+b^2}=-k(b-2a). \quad (4)$$

We take the negative sign since $\frac{b-2a}{x}$ is $-$ve.

$$\therefore ax+by=ax+\frac{b(a-2b)}{(b-2a)}x$$

$$=-2(a^2-ab+b^2)\frac{x}{b-2a}$$

$$=2k\sqrt{a^2-ab+b^2} \text{ from (4).}$$

Exercises XXVII.

1. Determine the maxima and minima (if any) of

(a) $x^5-5x^4+5x^3+10.$ (A.U. 47)

(b) $(x+5)^2(x^3-10).$ (B.A. Sub. 51)

(c) $x^3(x-2)^3.$ (B.A. Sub. 35)

(d) $x^3-18x^2+96x+4.$ (A.U. 45)

(e) $x+\frac{4}{x+2}.$ (B.A. Sub. 50)

(f) $(x-2)^3(x-3)^4.$ (B.Sc. 42 T.U.)

(g) $\frac{x}{1+x^2}.$ (B.Sc. 52 T.U.)

(h) $\dfrac{x^2 - 7x + 6}{x - 10}$. (B.Sc. 52 T.U.)

(i) $\dfrac{(x-1)^2 (x+2)}{(x-2)^2}$. (B.A. 54 M)

2. Find the maximum value of $\sin^2 x\,(1 + \cos x)^3$. (B.A. Sub. 40)

3. Investigate the maximum and minimum values of the function $y = \dfrac{1 - x + x^2}{1 + x + x^2}$. (B.A. Sub. 40)

4. Find the maximum and minimum radii vectors of the curve $\dfrac{c^2}{r^2} = \dfrac{a^2}{\sin^2\theta} + \dfrac{b^2}{\cos^2\theta}$. (B.A. Sub. 51)

5. Find the minimum value of $x \log x$.

6. Show that the maximum value of $\left(\dfrac{1}{x}\right)^x$ is $e^{1/e}$.

7. Find the maximum value of xe^{-x}.

8. If $xy\,(y - x) = 2a^3$, show that y has a minimum value when $x = a$.

9. If $f(x) = (x-a)^2 \phi(x)$, show that $f(x)$ attains its maximum or minimum at $x = a$ according as $\phi(a)$ is negative or positive.

10. (a) For the function $y = x^{2n-1}(a-x)^{2m}$ where m and n are positive integers, show that $x = a$ gives a minimum and $x = \dfrac{(2n-1)a}{2m+2n-1}$ gives a maximum, while $x = 0$ gives neither a maximum nor a minimum. (B.Sc. 51 T.U.)

(b) Show that if the number a is divided into two parts x and $(a - x)$, the greatest value of $x^m (a-x)^n$ is $\dfrac{m^m n^n a^{m+n}}{(m+n)^{m+n}}$. (B.Sc. 54 S)

11. If $a^2x^4 + b^2y^4 = c^4$, show that the maximum value of xy is $c^2/\sqrt{2ab}$. (B.Sc. 52 T.U.)

12. $f(x)$ is a cubic polynomial attaining maximum and minimum values when $x = -3$ and $x = 2$ respectively. If the maximum and minimum values are 10 and $-2{\cdot}5$ respectively, determine $f(x)$. (B.Sc. 52 T.U.)

13. A cubic function of x has turning values at $x = 1$ and $x = -\frac{5}{3}$; it vanishes when $x = 0$ and $x = 4$ when $x = 2$. Find the function. (Mys. U. 35)

14. Show that the function $\frac{\sin(x+a)}{\sin(x+b)}$ has no maximum or minimum, given that $b-a$ is not a multiple of π. Examine the case when $b-a$ is a multiple of π. (B.Sc. 53 T.U.)

15. Show that $\left(k-\frac{1}{k}-x\right)(4-3x^2)$ where k is a positive constant, has one and only one maximum value and only one minimum value. (B.Sc. 55 M)

Example.

The bending moment at B at a distance x from one end of a beam of length l uniformly loaded is given by the formula $M=\frac{1}{2}wlx-\frac{1}{2}wx^2$ where $w=$ load per unit length. Show that the maximum bending moment is at the centre of the beam.

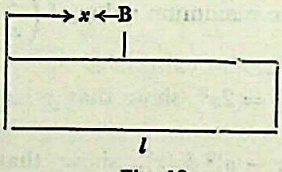

Fig. 12

$$M=\tfrac{1}{2}wlx-\tfrac{1}{2}wx^2.$$

M attains its maximum when $\frac{dM}{dx}=0$ and $\frac{d^2M}{dx^2}$ is $-$ve.

$$\frac{dM}{dx}=\tfrac{1}{2}wl-wx.$$

$$\frac{d^2M}{dx^2}=-w\ ;\ \frac{d^2M}{dx^2}\text{ is }-\text{ve}.$$

$\frac{dM}{dx}=0$ where $\frac{1}{2}wl-wx=0$, *i.e.*, when $x=\frac{1}{2}l$.

$\therefore$ The maximum bending moment is at the centre.

Exercises XXVIII.

1. The velocity of waves of length λ on deep water is proportional to $\sqrt{\frac{\lambda}{a}+\frac{a}{\lambda}}$, where a is a certain linear magnitude. Prove that the velocity is a minimum when $\lambda=a$.

2. If the total waste per mile in an electric conductor is $W=c^2r+\frac{t^2}{r}$, where c is the current in amperes, r resistance in

ohms per mile and t a constant depending on the interest in the investment and the depreciation of the plant, what is the relation between c, r and t when the waste is a minimum ?

3. The force exerted by a circular electrical current of radius a on a small magnet whose axis coincides with the axis of the circle varies as $\dfrac{x}{(a^2+x^2)^{5/2}}$ where x is a distance of the magnet from the plane of the circle. Prove that the force is a maximum when $x = \frac{1}{2} a$.

4. An adjustable lamp A is capable of moving only vertically and is placed right above the centre C of a horizontal circular table of diameter 8 feet. At what height should A be adjusted in order that the illumination at a point B one foot from the edge of the table be a maximum, given that the intensity of illumination at a point B is proportional to $\dfrac{\cos \angle CAB}{AB^2}$? (B.Sc. 42 T.U.)

5. (*a*) A battery of n cells with x cells in series per row and $\dfrac{n}{x}$ rows in parallel sends a current I through an external resistance R. Given that $I = \dfrac{xe}{\dfrac{x^2 r}{n} + R}$, where e is the E.M.F. and r the resistance, find the value of x for which the current is maximum. (B.A. Sub. 34)

(*b*) A battery contains 20 cells each of E.M.F. 1·8 volts and internal resistance 0·20 ohm. If x cells are arranged in series and $\dfrac{20}{x}$ rows in parallel, the current that the battery will send through an external resistance 0·36 ohm is given by $C = \dfrac{180x}{x^2+36}$. How many cells must be arranged in series to give the greatest possible current ? (B.A. Sub. 46)

6. A submarine telegraph cable consists of a core of copper wires with a covering made of non-conducting material. If x denotes the ratio of the radius of the core to the thickness of the covering, it is known that the speed of signalling varies as $x^2 \log \dfrac{1}{x}$. Show that the greatest speed is attained when $x = \dfrac{1}{\sqrt{e}}$. (B.A. Sub. 48)

7. The efficiency E of a screw is given by $E = \dfrac{t(1-\mu t)}{\mu + t}$ where t is the tangent of the angle of the screw, and μ the coefficient of friction. Find for what value of t the efficiency is a maximum.
(B.A. 41 ; B.Sc. 52 T.U

8. Assuming that the power given out by a voltaic cell is given by the formula $P = \dfrac{E^2 r}{(r+R)^2}$, where $E =$ constant electromotive force, $R =$ constant internal resistance, $r =$ external resistance, prove that P is a maximum when $r = R$.

9. The intensity of illumination at a point distant d feet from a light of candle power c is $\dfrac{c}{d^2}$. There are two lights P and Q 12 feet apart of candle powers 16 and 25 respectively. Find the point in PQ at which the intensity of illumination is a minimum.
(B.A. Sub. 3

10. The potential energy of a system is

$$\lambda^2 (\cos a - \sin a \cos \theta) + 2\mu^2 \sqrt{3 - 2\sin\theta}.$$

Find out if there is any turning value for the potential energy when θ lies in the interval 0 to $\dfrac{\pi}{2}$. If so, is it a maximum or a minimum ?
(B.A. 42 M

11. If a body of weight W is pushed by a horizontal force P lbs. up a plane inclined to the horizon at an angle a, the efficiency is given by $E = \dfrac{\tan a}{\tan(a+\lambda)}$, where λ is the angle of friction. If $\lambda = 14°$, find a when the efficiency is maximum and calculate the maximum value.

12. The angular distance θ of the bob of a pendulum at time t is given by $\theta = e^{-kt}\cos(nt + a)$. Prove that the angular distances become maximum at intervals of π/n.
(B.A. Sub. 5

§ 29·6. Almost in all the previous examples the relations between the dependent and independent variables are given. In the following examples, we have to find the relation between the variables from the given data and then find the maxima or minima as the case may be.

Examples.

Ex. 1. From a given circular sheet of metal it is required to cut out a sector so that the remainder can be formed into

a conical vessel of maximum capacity ; prove that the angle of the sector removed must be about 66°. (B.Sc. 41 T.U.)

Let θ be the angle of the sector ACB. Then $2\pi - \theta$ is the angle of the sector which has to be removed.

Length of arc $ACB = R\theta$ where R is radius of the circular disc.

Now the circumference of the base of the conical vessel must be the same as the length of the arc ACB.

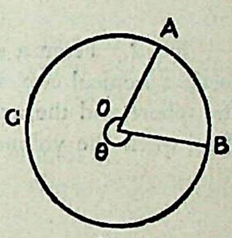

Fig. 13

Circumference of base $= R\theta$.

Radius of base $= \dfrac{R\theta}{2\pi}$.

Length of the slant height of the cone $= R$.

$\therefore$ Vertical height of the cone $= \left(R^2 - \dfrac{R^2\theta^2}{4\pi^2}\right)^{\frac{1}{2}}$.

Volume of the conical vessel $V = \dfrac{\pi}{3}\dfrac{R^2\theta^2}{4\pi^2}\left(R^2 - \dfrac{R^2\theta^2}{4\pi^2}\right)^{\frac{1}{2}}$;

$$V = \frac{R^3}{12\pi}\left(\theta^4 - \frac{\theta^6}{4\pi^2}\right)^{\frac{1}{2}}.$$

Now V will be the greatest when $\left(\theta^4 - \dfrac{\theta^6}{4\pi^2}\right)^{\frac{1}{2}}$ is greatest; that is when $\theta^4 - \dfrac{\theta^6}{4\pi^2}$ is a maximum.

Let $f(\theta) = \theta^4 - \dfrac{\theta^6}{4\pi^2}$.

$$f'(\theta) = 4\theta^3 - \frac{3}{2}\frac{\theta^5}{\pi^2}.$$

$$f''(\theta) = 12\theta^2 - \frac{15}{2}\frac{\theta^4}{\pi^2}.$$

$f(\theta)$ is a maximum when $4\theta^3 - \dfrac{3}{2}\dfrac{\theta^5}{\pi^2} = 0$

i.e., when $\theta = 0$ or $\theta = 2\pi\sqrt{\dfrac{2}{3}}$.

$\theta = 0$ is clearly inadmissible.

$\theta = 2\pi\sqrt{\dfrac{2}{3}}$ makes $f''(\theta)$ negative and gives the maximum value for $f(\theta)$.

The angle of the sector $= (2\pi - \theta)$ radians

$$= 2\pi - 2\pi\sqrt{\frac{2}{3}} \text{ radians}$$

$$= 1{\cdot}153 \text{ radians}$$

$$= 66° \, 6' \text{ approximately.}$$

Ex. 2. From a solid sphere, matter is scooped out so as to form a conical cup, with the vertex of the cup on the surface of the sphere and the axis passing through the centre of the sphere. Find when the volume of the cup is a maximum.

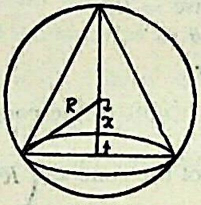

Fig. 14

Let x be the distance of the centre of the sphere from the base of the cone and let R be the radius of the sphere.

Height of the cone $= R + x.$

Radius of the base of the cone $= \sqrt{R^2 - x^2}.$

Volume of the cone $V = \frac{\pi}{3}(R + x)(R^2 - x^2).$

$$\frac{dV}{dx} = \frac{\pi}{3}(R^2 - 2Rx - 3x^2).$$

$$\frac{d^2V}{dx^2} = \frac{\pi}{3}(-2R - 6x).$$

V is a maximum when $\frac{dV}{dx} = 0$ and $\frac{d^2V}{d^2x}$ is —ve.

$\frac{dV}{dx} = 0$ when $R^2 - 2Rx - 3x^2 = 0$

i.e., when $x = -R$ or $\frac{R}{3}$.

$x = -R$ is clearly inadmissible ; $\frac{d^2V}{dx^2}$ for $x = \frac{R}{3}$ is negative.

$\therefore$ V is a maximum when $x = \frac{R}{3}$.

Height of cone $= R + x = \frac{4R}{3}.$

Radius of base $= \sqrt{R^2 - x^2} = \dfrac{2\sqrt{2}}{3} R.$

Volume of cone $= \dfrac{\pi}{3} \cdot \dfrac{4R}{3} \cdot \dfrac{8R^2}{9} = \dfrac{32\,\pi\, R^3}{81}.$

Ex. 3. Find the dimensions of a cylindrical vessel of greatest capacity which can be made from a given amount of sheet of metal (1) when the vessel has no lid and (2) when the vessel has a lid. (B.Sc. 53 T.U.)

(1) *When the vessel has no lid.*

Let S be the area of sheet metal used without lid.

S = surface area of vessel $= 2\pi xy + \pi x^2$, where x = radius of base and y = height.

Volume of vessel $= \pi x^2 y$.

$$V = \pi x^2 \left\{ \frac{S - \pi x^2}{2\pi x} \right\} = \frac{1}{2}(Sx - \pi x^3).$$

Then $\dfrac{dV}{dx} = \dfrac{1}{2}(S - 3\pi x^2)$; $\dfrac{d^2V}{dx^2} = -3\pi x.$

Now V is a maximum when $\dfrac{dV}{dx} = 0$ and $\dfrac{d^2V}{dx^2}$ is —ve.

$\dfrac{d^2V}{dx^2}$ is —ve when x is positive.

$$\frac{dV}{dx} = 0 \text{ when } x = \sqrt{\frac{S}{3\pi}}.$$

$\therefore$ V is a maximum when $x = \sqrt{\dfrac{S}{3\pi}}$

i.e., when $S = 3\pi x^2$.

$$S = 2\pi xy + \pi x^2.$$
$$3\pi x^2 = 2\pi xy + \pi x^2.$$
$$2\pi xy = 2\pi x^2.$$
$$\therefore y = x.$$

$\therefore$ Height of the vessel $= \sqrt{\dfrac{S}{3\pi}}.$

(2) *With lid.*

S = surface area of the vessel $= 2\pi xy + 2\pi x^2$, where x = radius of base and y = height.

V, volume of vessel $= \pi x^2 y = \dfrac{\pi x^2 (S - 2\pi x^2)}{2\pi x}$

$$= \frac{1}{2}(Sx - 2\pi x^3).$$

Then $\dfrac{dV}{dx} = \dfrac{1}{2}(S - 6\pi x^2)$; $\dfrac{d^2V}{dx^2} = -6\pi x.$

Now V is a maximum when $\frac{dV}{dx}=0$ and $\frac{d^2V}{dx^2}$ is —ve.

$\frac{dV}{dx}=0$ when $S-6\pi x^2=0$, *i.e.*, $x=\sqrt{\frac{S}{6\pi}}$.

For this value of x, $\frac{d^2V}{dx^2}$ is —ve.

For this value of x, V is a maximum.

$$S=2\pi xy+2\pi x^2.$$
$$6\pi x^2=2\pi xy+2\pi x^2.$$
$$y=2x=2\sqrt{\frac{S}{6\pi}}.$$

$\therefore$ Height of the vessel $=2\sqrt{\frac{S}{6\pi}}$.

Ex. 4. The cost of fuel in running an engine is proportional to the square of the speed and is Rs. 48 per hour for a speed of 16 m.p.h. Other costs amount to Rs. 300 per hour. What is the most economical speed ? (B.Sc. Anc. 63)

If v m.p.h. is the speed and Rs. c is the cost of fuel, $c=kv^2$, where k is a constant.

When $v=16$, $c=48$. $\therefore$ $k=\frac{3}{16}$.

Hence the cost of fuel $=\frac{3}{16}v^2$.

Total running cost per hour = Rs. $\left(300+\frac{3}{16}v^2\right)$.

If the distance travelled is s miles, the number of hours taken up is $\frac{s}{v}$ hours.

$\therefore$ Total cost for the journey is Rs. y,

where $y=\frac{s}{v}\left(300+\frac{3}{16}v^2\right)$.

For the most economical speed,

$$\frac{dy}{dv}=0 \text{ and } \frac{d^2y}{dv^2} \text{ is +ve.}$$
$$\frac{dy}{dv}=s\left(-\frac{300}{v^2}+\frac{3}{16}\right)=0. \quad \therefore \quad v=40.$$
$$\frac{d^2y}{dv^2}=\frac{600s}{v^3}>0.$$

Hence the most economical speed per hour is 40 m.p.h.

Exercises XXIX.

1. A rectangular sheet of metal has four equal square portions removed at the corners and the sides are then turned up so as to form a rectangular box. Show that when the volume contained in the box a maximum, the depth will be

$$\frac{1}{6}\left\{a+b-\sqrt{a^2-ab+b^2}\right\},$$

where a and b are the sides of the original rectangle.

(B.Sc. 52 M)

2. Show that, if the sum of the lengths of the hypotenuse and another side of a right-angled triangle is given, its area is a maximum when the angle between those sides is 60°.

(B.Sc. 51 T.U. ; B.A. 38 M)

3. A window is in the shape of a rectangle surmounted by a semi-circle. If the perimeter of the window be a fixed length l, find the maximum area. (B.A. Sub. 37)

4. Show that a conical tent of given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base. (B.Sc. 50 M)

5. A wire of length a is cut into two parts which are bent respectively in the form of a square and a circle. Show that the least value of the sum of the areas so formed is $\dfrac{a^2}{4(\pi+4)}$.

6. Through the point (2, 3) a straight line is drawn making positive intercepts on the axes. Show that the area of the triangle thus formed is least when the ratio of the intercepts on the x and the y axes is 2 : 3. (B.A. Sub. 42)

7. (*a*) Show that the maximum rectangle inscribed in a circle is a square. (B.A. 50 M)

(*b*) Show that the area of the greatest rectangle inscribed in a given ellipse and having its sides parallel to the axes of the ellipse is $2ab$. (B.Sc. 51 T.U.)

(*c*) Find the dimensions of the largest rectangle that can be inscribed in $x^{2/3}+y^{2/3}=a^{2/3}$. (B.Sc. 65 M)

8. (*a*) If PQ is a double ordinate of $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and A be an end of the major axis, find the maximum area of the triangle APQ. (B.A. 46 M)

(*b*) Prove that the least intercept made by the axes on a tangent to the ellipse is equal to the sum of the semi-axes of the ellipse. (B.Sc. 51 T.U.)

(*Vide* Ex. 3, Page 107.)

Show further that, in the above case, the tangent is divided at the point of contact into two portions equal to semi-axes of the ellipse. (B.Sc. 52 M.)

9. A, O, B are fixed collinear points in the given order. A pair of variable circles through A, O and B, O have their diameters OC and OD at right angles. Show that the least length of CD is AB. (B.A. 46 M.)

10. A vessel is in the form of a right circular cylinder with a hemispherical top and its base is twice as thick as the rest of its surface. Show that for a given volume, the weight of the vessel is least when its height is three times its base radius. (B.A. Sub. 41 M.)

11. Prove that the volume of the greatest right circular cone that can be inscribed in a given sphere is $\frac{8}{27}$ of the volume of the sphere. (B.A. Sub. 43 ; 50 A.U.)

12. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a is $\frac{2a}{\sqrt{3}}$. (B.Sc. 38 M.)

13. Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle a is $\frac{4}{27}\pi h^3 \tan^2 a$. (B.A. 38 M ; B.Sc. 51 M.)

14. The surface area of a cone including the base is 4π square feet. Find the dimensions of the cone when the volume enclosed is a maximum. (B.A. 39 M.)

15. An open tank is to be constructed with a square base and vertical sides to hold a given quantity of water. Show that the expense of lining the tank with lead will be the least if the depth be half the width. (B.A. Sub. 38)

16. A thin closed rectangular box is to have one edge n times the length of another edge and the volume of the box is given to be V. Prove that the least surface S is given by

$$nS^3 = 54(n+1)^2 V^2. \qquad \text{(B.Sc. 43 T.U.)}$$

17. A right circular cylinder closed at both ends has a fixed total surface area S although the radius r and the height h vary. Prove that, as the radius increases, the volume will also increase until the diameter becomes equal to the height of the cylinder. If V is the maximum volume attained, show that $54\pi V^2 = S^3$.

18. The sum of the surfaces of a sphere and a cube is given. Show that when the sum of their volumes is least, the diameter of the sphere is equal to the edge of the cube.

19. Given the total surface of a cone, show that when the volume of the cone is a maximum, the semi-vertical angle will be $\sin^{-1}(1/3)$.

Given the volume of the cone, show that the total surface will be a minimum for the same value of the vertical angle.

20. A cone is circumscribed about a sphere of radius R; show that when the volume of the cone is a minimum, its altitude is $4R$ and its semi-vertical angle is $\sin^{-1}(1/3)$. (B.Sc. 40 T.U.)

21. Into a full conical wine glass of depth a and generating angle α, there is carefully dropped a sphere of such size as to cause the greatest overflow. Show that the radius of the sphere is $\dfrac{a \sin \alpha}{\sin \alpha + \cos 2\alpha}$.

22. The distance between the centres of two spheres of radii a and b is c, $(c > a + b)$. Find the point P on the line joining the centres of the spheres such that the sum of the surfaces of both spheres visible from P may be a maximum. (B.Sc. 32 M)

23. Show that, when the curved surface of a cylinder inscribed in a sphere of radius R is a maximum, the altitude is $R\sqrt{2}$. (B.Sc. 51 T.U.)

Show that when the whole surface is a maximum that surface is to the surface of the sphere in the ratio of $\sqrt{5} + 1$ to 4.

24. A cylinder is inscribed in a cone. Show that the curved surface is a maximum when the altitude is half that of the cone.

Show also that the total surface cannot have a maximum unless the semi-vertical angle of the cone is less than $\tan^{-1}(1/2)$.

25. Show that the semi-vertical angle of the cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

26. A lane runs at right angles out of a road 18 feet wide. Find to one decimal place how many feet wide the lane must be if it is just possible to carry a pole 45 feet long from the road into the lane, keeping it horizontal. (B.Sc. 39 M)

27. The strength of a rectangular beam varies as the produ of the breadth and the square of the depth. Find the bread and depth of the stiffest rectangular beam that can be cut fro a cylindrical log the radius of whose cross section is a inches.

(B.Sc. 38 M ; B.A. 51 M

28. A rectangular box of fixed volume V is to be twice as lo as it is wide. The material in the top and four sides costs thr times as much per square foot as that in the bottom. What a the most economical proportions ? (B.Sc. 35 M ; B.A. 52 M

29. Given a line l and two fixed points A, B on the same si of it, show that the least value of $AP + PB$ (where P is any po on l) occurs when AP and PB make equal angles with l. G a geometrical construction for finding P so that AP and PB ma equal angles with l (neither A nor B lies on l). (B.A. 45 M

30. A circular sector has a given perimeter ; show that wh the area is a maximum, the arc is double the radius and that t maximum area is equal to the square on the radius.

(B.Sc. 53 Os.U

31. The section of a tunnel is a rectangle surmounted a semi-circle on its longer side. Given that the perimeter of section is 20 feet, find its dimensions in order that its area m be a maximum. (B.Sc. 52 T.U

32. Prove that in the cardioid $r = a\ (1 + \cos\theta)$, t maximum distance of any point of the curve from the axis $\dfrac{3\sqrt{3}a}{4}$. (B.Sc. 52 Os.U

33. Prove that the greatest distance of a normal to an ellip from its centre is the difference of its semi-axes. (B.Sc. 53

34. A given quantity of metal is cast into the form of a b cylinder with rectangular base and semi-circular ends. Find t ratio of the length of the rectangular base to the diameter of t semi-circular ends when the total surface area is least. (Tr.U.

35. From a fixed point P on the circumference circle of radius a, the perpendicular PR is drawn to tangent at Q. Prove that the maximum area of ΔPQR $3\sqrt{3}a^2/8$.

36. The regulations for parcel post require that the sum the length and girth of a parcel must not exceed 6 ft. Prove t the right circular cylinder of greatest volume which can be s is 2 ft. long and 4 ft. in girth.

37. The corner of a rectangular sheet of paper is turned down just to reach the other edge of the page; find when the length of the crease is a minimum; also when the area of the part turned down is a minimum.

§ 30. Concavity and convexity, points of inflexion.

If in the neighbourhood of a point P on a curve, the curve is above the tangent at P [as in figures (a) and (b)], it is said to be *concave upwards*; if the curve is below the tangent at P [as in figures (c) and (d)], it is said to be *concave downwards* or *convex upwards*.

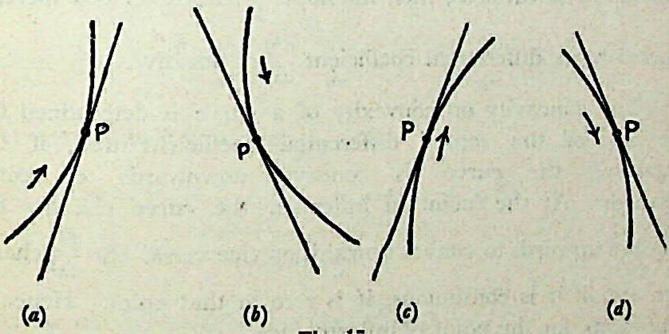

Fig. 15

If at a point P, a curve changes its concavity from upwards to downwards or vice versa [as in figures (e) and (f)], P is called a *point of inflexion*.

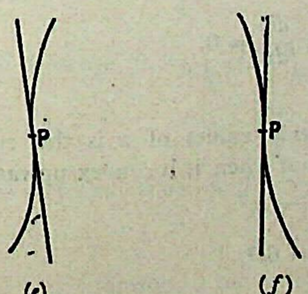

Fig. 15

From this definition, it is seen that the curve crosses its tangent at the point of inflexion and that a point of inflexion lies between a maximum and a minimum.

As the point on the curve in figures (*a*) and (*b*) moves to the right (the direction of arrows), the tangent turns about its point of contact anticlockwise and therefore the angle which it makes with the x-axis increases. So we get at all points in the neighbourhood of P on the curve when it is concave upwards ; the slope of the curve, *i.e.*, $\frac{dy}{dx}$ increases as x increases. Therefore its differential coefficient is positive, *i.e.*, $\frac{d^2y}{dx^2}$ is positive.

Similarly, if at all points in the neighbourhood of P, the curve is concave downwards, then the slope $\frac{dy}{dx}$ decreases as x increases. Therefore its differential coefficient $\frac{d^2y}{dx^2}$ is negative.

The concavity or convexity of a curve is determined from the sign of the second differential coefficient and, if it is negative, the curve is concave downwards or convex upwards. At the point of inflexion, the curve changes from concave upwards to convex upwards or vice versa. So $\frac{d^2y}{dx^2}$ changes sign and if it is continuous, it is zero at that point. Hence the conditions for the point of inflexion are

(1) $\frac{d^2y}{dx^2}=0$ at that point.

(2) $\frac{d^2y}{dx^2}$ changes its sign as x increases through the values at which $\frac{d^2y}{dx^2}=0$, *i.e.*, $\frac{d^3y}{dx^3}\neq 0$.

Examples.

Ex. 1. For what values of x is the curve $y=3x^2-2x^3$ concave upwards and when is it convex upwards ?

$$y=3x^2-2x^3.$$

Then $\frac{dy}{dx}=6x-6x^2$.

$$\frac{d^2y}{dx^2}=6-12x=-6\,(2x-1).$$

If $x>\frac{1}{2}$, $\frac{d^2y}{dx^2}$ is negative and so convex upwards.

If $x < \frac{1}{2}$, $\frac{d^2y}{dx^2}$ is positive and so concave upwards.

If $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} = -12$ and so there is a point of inflexion at $x = \frac{1}{2}$, *i.e.*, at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Ex. 2. Find the points of inflexion on the cubic $y = \frac{a^2x}{x^2 + a^2}$ and show that they lie on a straight line. (B.A. 42 M)

$$y = \frac{a^2x}{x^2 + a^2}.$$

Then $\frac{dy}{dx} = \frac{a^2(a^2 - x^2)}{(x^2 + a^2)^2}$ and $\frac{d^2y}{dx^2} = \frac{-2a^2x(3a^2 - x^2)}{(x^2 + a^2)^3}$.

At the points of inflexion, $\frac{d^2y}{dx^2} = 0$.

$\therefore$ $x(3a^2 - x^2) = 0$. $\therefore$ $x = 0$ or $\pm\sqrt{3}a$.

$$\frac{d^3y}{dx^3} = \frac{-6a^2(x^4 + a^4 - 6a^2x^2)}{(x^2 + a^2)^4}.$$

At the points $x = 0$ or $\pm\sqrt{3}a$, $\frac{d^3y}{dx^3} \neq 0$.

When $x = 0$, $y = 0$; $x = \sqrt{3}a$, $y = \frac{\sqrt{3}}{4a}$.

$x = -\sqrt{3}a$; $y = -\frac{\sqrt{3}a}{4}$.

The points of inflexion are $(0, 0)$, $\left(\sqrt{3}a, \frac{\sqrt{3}}{4}a\right)$,

$$\left(-\sqrt{3}a, \frac{-\sqrt{3}}{4}a\right).$$

These three points of inflexion lie on the straight line $x = 4y$.

Exercises XXX.

1. Find the points of inflexion in the following cases :—

(1) $y = x^4 - 6x^2 + 8x - 1$. (4) $y = a\sin x + b\cos x$.

(2) $y = x^3 - 9x^2 + 7x - 6$. (5) $y = a\cos^2 x + b\sin^2 x$.

(3) $y = \cos x$. (6) $y = \frac{x^3}{a^2 + x^2}$.

(7) $xy^2 = a^2(a - x)$. (9) $y = \dfrac{\log x}{x^{1/3}}$, $(0 < x)$.

(8) $y = x^3 e^{-x}$. (B.Sc. 43 M) (B.Sc. 44 M

2. Show that the curve $y = k \sin x$ cuts the x-axis at inflexional points.

3. Find the points of inflexion on the curve $y = (x - a)(x - b)(x - c)$. (B.Sc. 46 M

4. Show that the curve $y = \dfrac{6x}{x^2 + 3}$ has 3 points of inflexion. (B.A. 40 M

5. Show that the curve $y = \dfrac{1 - x}{1 + x^2}$ has three points of inflexion and show further that they lie on a straight line and find its equation.

6. Prove that the points of inflexion on the curve $x^2y + a^2(x + y) = a^3$ lie on the straight line $x + 4y = 3a$. (B.Sc. 44 M

7. Determine the relation between the constants a, b, c so that the curve $y = ax^3 + bx^2 + cx + d$ may have a point of inflexion for $x = 1$.

8. Prove that the curve $y = x^4 - 2x^3 - 36x^2 + 7$ is everywhere concave upwards except between the ordinates $x = -$ and $x = 3$.

9. If $\left(p + \dfrac{a}{v^2}\right)(v - b) = c$, ($a, b, c$ being constants), show by expressing p as a function of v that when $v = 3b$, the (p, v) curve has a point of inflexion where the tangent is parallel to the v-axis. Determine the value of c in that case. (B.A. Sub.

10. Find the turning points and points of inflexion on the curve $a^2y^2 = x^2(a^2 - x^2)$. Also trace the curve. (B.Sc. 5

CHAPTER VII

EXPANSION OF FUNCTIONS

§ 31·1. Infinite series.

The infinite series $u_1 + u_2 + u_3 + \ldots\ldots + u_n + \ldots\ldots$ to ∞ is usually denoted by $\sum_1^\infty u_n$ or simply $\sum u_n$. The sum of the first n terms of the series, *viz.*, $u_1 + u_2 + \ldots\ldots + u_n$ is denoted by S_n. There are three alternatives.

(1) S_n may tend to a finite limit as n tends to infinity.

(2) It may tend to infinity as $n \to \infty$.

(3) It may tend to more than one limit as $n \to \infty$.

In the first case, when S_n tends to a finite limit as n tends to infinity, the series $\sum u_n$ is said to be *convergent* and the finite limit is called its sum to infinity. In the second case, when S_n tends to infinity as n tends to infinity, the series is said to be *divergent*. In the third case when S_n tends to more than one limit as n tends to infinity, it is said to be an *oscillating* series.

For example, consider the geometric progression

$$1 + x + x^2 + \ldots\ldots + x^n + \ldots\ldots$$

$$S_n = \frac{1 - x^n}{1 - x}.$$

If $|x| < 1$, then $\underset{n\to\infty}{\text{Lt}}\ S_n = \dfrac{1}{1 - x}$.

$\therefore$ This geometric progression is convergent for $|x| < 1$.

If $|x| > 1$, then $\underset{n\to\infty}{\text{Lt}}\ S_n = \infty$.

$\therefore$ For these values of x, this series is divergent.

If $x = 1$, then $S_n = 1 + 1 + \ldots\ldots + 1$
$= n$.

$\therefore$ $\underset{n\to\infty}{\text{Lt}}\ S_n$ is infinity.

Here also the geometric series is divergent.

If $x = -1$, the series becomes

$$1 - 1 + 1 - 1 + 1 - 1 + \ldots\ldots$$

For this series $S_n = 0$ if n is even and
$S_n = 1$ if n is odd.

In this case, the series is oscillatory. If a series is diverge or oscillatory, it does not possess a 'sum to infinity' as defin above. The phrase 'sum to infinity' is applicable only a convergent series.

If the sum to n terms, S_n, can be expressed by elementa functions, the nature of the series can be determined by findi whether the expression tends to a finite limit or to infinity or more than one limit when $n \to \infty$. But it is not always possi or convenient to find the sum S_n. There are various tests to fi whether a series is convergent or not. For those tests, the studen referred to any standard text-book in Algebra. For our purpose is enough to know that if $u_1 + u_2 + \ldots\ldots u_n + \ldots\ldots$ is converge and if it is possible to find S such that $\underset{n\to\infty}{\text{Lt}} | S - S_n | = 0$, th S is the sum to infinity of the infinite series $\Sigma\, u_n$.

§ 31·2. **Taylor's Theorem.** In the fifth chapter, we ha proved the generalised Mean Value Theorem

$$f(x) = f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2\,!} f''(a) + \ldots\ldots + \frac{(x-a)^n}{n\,!} f^n(a + \overline{x-a}\ldots$$

In this equation, let us assume the sum of the first n terms $S_n(x)$ and the last term as $R_n(x)$.

$$\therefore\ f(x) = S_n(x) + R_n(x).$$

$R_n(x) = \dfrac{(x-a)^n}{n\,!} f^n(a + \overline{x-a}\,\theta)$. This is called Lagrang form of the remainder.

§ 31·3. **Cauchy's form of Remainder.**

Let
$$F(x) = f(b) - f(x) - (b-x)\,f'(x) - \frac{(b-x)^2}{2\,!} f''(x) \ldots\ldots$$
$$\ldots\ldots - \frac{(b-x)^{n-1}}{(n-1)\,!} f^{n-1}(x) - (b-x)\ldots$$

where R is given by

$$0 = f(b) - f(a) - (b-a)\,f'(a) - \frac{(b-a)^2}{2\,!} f''(a) \ldots\ldots - (b-a)\ldots$$

Then $F(a) = 0$ and $F(b) = 0$.

$\therefore\ F'(x)$ vanishes for at least one value of between a and b, say, for x_1.

$$\therefore\ F'(x_1) = 0.$$

Differentiating $F(x)$, we get

$$F'(x) = -f'(x) + f'(x) - (b-x)f''(x) + (b-x)f''(x) \ldots\ldots - \frac{(b-x)^{n-1}}{(n-1)!}f^n(x) + R$$

$$= R - \frac{(b-x)^{n-1}}{(n-1)!}f^n(x).$$

$$F'(x_1) = 0. \quad \therefore \quad R = \frac{(b-x_1)^{n-1}}{(n-1)!}f^n(x_1).$$

Writing $a + \theta(b-a)$ for x_1 where $0 < \theta < 1$, we get $b - x_1 = b - a - \theta(b-a) = (b-a)(1-\theta)$.

$$\therefore \quad f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \ldots\ldots + \ldots\ldots + \frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(b-a)^n}{(n-1)!}(1-\theta)^{n-1}f^n(a + \theta\,\overline{b-a}).$$

Here writing x for b, we get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots\ldots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(x-a)^n(1-\theta)^{n-1}}{(n-1)!}f^n(a + \overline{x-a}\,\theta).$$

Here $R_n(x) = \dfrac{(x-a)^n(1-\theta)^{n-1}}{(n-1)!}f^n(a + \overline{x-a}\,\theta)$

where $0 < \theta < 1$.

This is called *Cauchy's form of Remainder.*

§ 31·4. Taylor's and Maclaurin's series.

$$f(x) = S_n(x) + R_n(x).$$

$$\therefore \quad f(x) - S_n(x) = R_n(x).$$

$$\operatorname*{Lt}_{n\to\infty}\{f(x) - S_n(x)\} = \operatorname*{Lt}_{n\to\infty} R_n(x).$$

If limit of $R_n(x)$ is zero,

then $\operatorname*{Lt}_{n\to\infty}\{f(x) - S_n(x)\} = 0$.

$\therefore$ As $n\to\infty$, $f(x)$ becomes the infinite series.

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots\ldots \text{ to } \infty.$$

This is called *Taylor's series.*

Here if we put $a = 0$, we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots$$
$$\ldots\ldots + \frac{x^n}{n!}f^n(0) + \ldots\ldots \text{ to } \infty$$

This is called Maclaurin's series. This series is useful in finding the expansion of functions.

Examples.

Ex. 1. Expand $\sin x$ as an infinite series.

Let $f(x)$ be $\sin x$.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots$$

Here $R_n(x)$ must tend to zero.

$$f(x) = \sin x \qquad \therefore\ f(0) = 0.$$
$$f'(x) = \cos x \qquad \therefore\ f'(0) = 1.$$
$$f''(x) = -\sin x \qquad \therefore\ f''(0) = 0.$$
$$f'''(x) = -\cos x \qquad \therefore\ f'''(0) = -1.$$

More generally, $f^n(x) = \sin\left(x + \frac{n\pi}{2}\right)$.

$$R_n(x) = \frac{x^n}{n!}f^n(\theta x)$$
$$= \frac{x^n}{n!}\sin\left(\theta x + \frac{n\pi}{2}\right).$$

$$\sin\left(\theta x + \frac{n\pi}{2}\right) \ngtr 1 \text{ and } \underset{n\to\infty}{\mathrm{Lt}}\ \frac{x^n}{n!} = 0.$$

$$\therefore\ \underset{n\to\infty}{\mathrm{Lt}}\ R_n(x) = 0.$$

$$\therefore\ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots$$

Similarly, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots\ldots$

Ex. 2. Expand e^x as an infinite series.

Let $f(x)$ be e^x.

$$f(x) = e^x \qquad f(0) = 1.$$
$$f'(x) = e^x \qquad f'(0) = 1.$$
$$f''(x) = e^x \qquad f''(0) = 1.$$
$$f^n(x) = e^x. \qquad f^n(\theta x) = e^{\theta x}.$$

$$R_n(x) = \frac{x^n}{n!}e^{\theta x}.$$

$$\underset{n\to\infty}{\text{Lt}} R_n(x) = e^{\theta x} \underset{n\to\infty}{\text{Lt}} \frac{x^n}{n!} = 0.$$

$$\therefore \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \frac{x^n}{n!} + \dots \infty.$$

Ex. 3. Expand $\log_e(1+x)$.

Let $f(x)$ be $\log_e(1+x)$.

$$f(x) = \log(1+x) \qquad f(0) = 0.$$

$$f'(x) = \frac{1}{1+x} \qquad f'(0) = 1.$$

$$f''(x) = \frac{-1}{(1+x)^2} \qquad f''(0) = -1.$$

$$f'''(x) = \frac{1\cdot 2}{(1+x)^3} \qquad f'''(0) = 1\cdot 2$$

$$f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} \quad \therefore f^n(\theta x) = \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n}.$$

$$\therefore R_n(x) = \frac{x^n}{n!} \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n}$$

$$= (-1)^{n-1}\frac{1}{n}\cdot\left(\frac{x}{1+\theta x}\right)^n.$$

If x is positive and is ≤ 1, $\frac{x}{1+\theta x} < 1$.

$$\therefore \underset{n\to\infty}{\text{Lt}} \left(\frac{x}{1+\theta x}\right)^n = 0. \quad \text{Again } \underset{n\to\infty}{\text{Lt}} \frac{1}{n} = 0.$$

$$\therefore \underset{n\to\infty}{\text{Lt}} R_n(x) = 0.$$

If x is negative, $\frac{x}{1+\theta x}$ need not be less than 1.

$\therefore$ The previous proof is not valid.

Cauchy's form of remainder is

$$R_n(x) = (-1)^{n-1} \frac{x^n (1-\theta)^{n-1}(n-1)!}{(n-1)!(1+\theta x)^n}$$

$$= (-1)^{n-1} \frac{x^n}{1+\theta x}\cdot\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}.$$

If $|x| < 1$, $1-\theta < 1+\theta x$.

$$\therefore \frac{1-\theta}{1+\theta x} < 1.$$

$$\therefore \underset{n\to\infty}{\text{Lt}} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} = 0.$$

Also $x^n \to 0$, if $|x| < 1$.

$1 + \theta x$ is constant as far as n is concerned.

$$\therefore \operatorname*{Lt}_{n\to\infty} R_n(x) = 0.$$

$\therefore$ If $|x| < 1$,

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \ldots\ldots$$

Ex. 4. Expand $(1+x)^m$.

Let $f(x)$ be $(1+x)^m$.

$$f(x) = (1+x)^m, \qquad f(0) = 1.$$
$$f'(x) = m(1+x)^{m-1}, \qquad f'(0) = m.$$
$$f''(x) = m(m-1)(1+x)^{m-2}, \qquad f''(0) = m(m-1).$$
$$f^n(x) = m(m-1)\ldots\ldots(m-n+1)(1+x)^{m-n}.$$
$$\therefore f^n(\theta x) = m(m-1)\ldots\ldots(m-n+1)(1+\theta x)^{m-n}.$$

If m is a positive integer, the series for $f(x)$ will contain $(m+$ terms since when $n > m$, $f^n(x) = 0$.

If m is not a positive integer, we have to find the limit $R_n(x)$ as $n \to \infty$. Using Cauchy's form of remainder,

$$R_n(x) = \frac{m(m-1)\ldots\ldots(m-n+1)}{(n-1)!} x^n (1-\theta)^{n-1}(1+\theta x)^{\ldots}$$

The infinite series $1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots\ldots$ is diverg for $|x| > 1$. Therefore it is enough if we consider the resul values of x where $|x| < 1$.

$$R_n(x) = mx(1+\theta x)^{m-1}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \times$$
$$\frac{(m-1)(m-2)\ldots\ldots(m-n+1)}{(n-1)!}x^{\ldots}$$

When $|x| < 1$, $\frac{1-\theta}{1+\theta x} < 1$.

$$\therefore \operatorname*{Lt}_{n\to\infty}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} = 0$$

and also $mx(1+\theta x)^{m-1}$ is a finite quantity.

$\frac{(m-1)(m-2)\ldots\ldots(m-n+1)}{(n-1)!}x^{n-1}$ is the nth term the convergent series

$$1 + (m-1)x + \frac{(m-1)(m-2)}{2!}x^2 + \ldots$$

$\therefore$ It is zero.

$$\therefore \operatorname*{Lt}_{n\to\infty} R_n(x) = 0.$$

$$\therefore\ (1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \dots$$

Here $|x| < 1$.

Ex. 5. Show that $e^x \cos x = 1 + 2\cos\frac{\pi}{4}x + 2^{2/2}\cos\frac{2\pi}{4}\cdot\frac{x^2}{2!}$

$$+ 2^{3/2}\cdot\cos\frac{3\pi}{4}\cdot\frac{x^3}{3!} + \dots + 2^{n/2}\cos\frac{n\pi}{4}\cdot\frac{x^n}{n!} + \dots$$

Let $f(x)$ be $e^x \cos x$.

Then $f(x) = e^x \cdot \dfrac{e^{ix} + e^{-ix}}{2}$

$$= \frac{e^{x(1+i)} + e^{x(1-i)}}{2}.$$

$$\therefore\ f^n(x) = \frac{(1+i)^n e^{x(1+i)} + (1-i)^n e^{x(1-i)}}{2}.$$

$$\therefore\ f^n(0) = \frac{(1+i)^n + (1-i)^n}{2} = 2^{n/2}\cos\frac{n\pi}{4}.$$

$$\therefore\ e^x \cos x = f(0) + x\cdot f'(0) + \frac{x^2}{2!}f''(0) + \dots$$

$$\dots + \frac{x^n}{n!}f^n(0) + \dots$$

$$= 1 + x\cdot 2^{1/2}\cos\frac{\pi}{4} + \frac{x^2}{2!}\cdot 2^{2/2}\cdot\cos\frac{2\pi}{4} + \dots$$

$$\dots + \frac{x^n}{n!}2^{n/2}\cdot\cos\frac{n\pi}{4} + \dots$$

§ 32. From the differential coefficients of certain functions, we can expand those functions as power series. There are certain conditions to be satisfied by those functions. We are not giving those conditions here, but only the method of expansion.

Ex. 6. If $|x| < 1$, expand $\tan^{-1}x$ as a series.

Let $f(x) = \tan^{-1}x = a_0 + a_1x + a_2x^2 + \dots$

Then $\quad f'(x) = \dfrac{1}{1+x^2} = a_1 + 2a_2x + 3a_3x^2 + \dots$

Also $(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \dots$

Comparing the two series, we get

$$a_2 = a_4 = a_6 = a_8 = \dots = 0$$

$$a_1 = 1,\ 3a_3 = -1,\ 5a_5 = 1 \dots$$

Also $\tan^{-1}(0) = a_0$. Taking the principal value alone, we get $a_0 = 0$.

$\therefore \tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 \ldots$ to ∞.

$(1 + x^2)^{-1}$ can be expanded by binomial theorem only when $|x| < 1$.

The series for $\tan^{-1} x$ is called *Gregory's Series*.

Ex. 7. Find the expansion of $\sin^{-1} x$.

Let $f(x) = \sin^{-1} x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

$$\therefore f'(x) = \frac{1}{\sqrt{1 - x^2}} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

$$\text{But } \frac{1}{\sqrt{1 - x^2}} = (1 - x^2)^{-1/2} = 1 + \tfrac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \ldots$$

if $x^2 < 1$.

Comparing the two series, we get

$$a_2 = a_4 = a_6 = \ldots 0.$$

$$a_1 = 1,\ 3a_3 = \tfrac{1}{2},\ 5a_5 = \frac{1 \cdot 3}{2 \cdot 4}, \ldots$$

Also $\sin^{-1} 0 = a_0$. Taking the principal value only, we get $a_0 = 0$.

$$\therefore \sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \ldots$$

§ 33. Using differential equations, we can expand certain functions. The following examples will illustrate the method:

Ex. 8. Expand $\tan^{-1} x$ (*vide* Ex. 6).

Let $y = \tan^{-1} x$.

$$\therefore y_1 = \frac{1}{1 + x^2}.$$

$$\therefore (1 + x^2)\, y_1 = 1.$$

Differentiating this equation n times, we get

$$(1 + x^2)\, y_{n+1} + 2nxy_n + n(n - 1)\, y_{n-1} = 0.$$

If we put $x = 0$ in this equation, we get

$$(y_{n+1})_{x=0} = -n(n - 1)\,(y_{n-1})_{x=0}. \quad (1)$$

$$\text{Also } (y)_{x=0} = 0,\ (y_1)_{x=0} = 1.$$

Substituting the values 2, 3, 4, 5, ... for n in equation (1) we get

$(y_3)_{x=0} = -2 \cdot 1\,(y_1)_0 = -2\,!$; $(y_2)_{x=0} = 0$;

$(y_5)_{x=0} = -4 \cdot 3\,(y_3)_0 = 4\,!$; $(y_4)_{x=0} = -3 \cdot 2\,(y_2)_{x=0} = 0$;

$(y_7)_{x=0} = -6 \cdot 5\,(y_5)_{x=0} = -6\,!$; $(y_6)_{x=0} = 0$.

All the even coefficients vanish.

$$\therefore \quad \tan^{-1} x = x - 2\,!\frac{x^3}{3\,!} + 4\,!\,.\frac{x^5}{5\,!} \ldots$$
$$= x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5.$$

Ex. 9. Expand $\cos(m \sin^{-1} x)$ as a power series.

Let $y = \cos(m \sin^{-1} x) = a_0 + a_1 x + a_2 \dfrac{x^2}{2\,!} + a_3 \cdot \dfrac{x^3}{3\,!} + \ldots$ (1)

$$a_0 = (y)_{x=0} = \cos(m \sin^{-1} 0) = 1.$$

Differentiating y, we get

$$y_1 = -\sin(m \sin^{-1} x)\,.\,\frac{m}{\sqrt{1-x^2}}. \qquad (2)$$

$$\therefore \quad y_1^2(1-x^2) = m^2 \sin^2(m \sin^{-1} x)$$
$$= m^2(1-y^2).$$

Differentiating this equation and cancelling the common factor $2y_1$, we get

$$(1-x^2)\,y_2 - xy_1 + m^2 y = 0. \qquad (3)$$

Differentiating this equation (3) n times, we get

$$(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} + (m^2 - n^2)\,y_n = 0. \qquad (4)$$

Substituting the value of $x = 0$ in (4), we get

$$(y_{n+2})_{x=0} = -(m^2 + n^2)\,(y_n)_{x=0}. \qquad (5)$$

Differentiating equation (1) $n+2$ times, we get

$$y_{n+2} = a_{n+2} + a_{n+3}\,x + \ldots$$
$$\therefore \quad (y_{n+2})_{x=0} = a_{n+2}.$$

Similarly $(y_n)_{x=0} = a_n,$

$$(y_{n-1})_{x=0} = a_{n-1}.$$
$$(y_1)_{x=0} = a_1.$$

From (5), we get $a_{n+2} = -(m^2 - n^2)\,a_n.$ (6)

From (2), $a_1 = 0.$

From (3), $a_2 = -m^2.$

From (6), $a_3 = -(m^2 - n^2)\,a_1 = 0.$

Similarly $a_1 = a_3 = a_5 = a_7 = \ldots = 0.$

$$\therefore \quad a_4 = -(m^2 - 2^2)\,a_2 = m^2(m^2 - 2^2).$$
$$a_6 = -(m^2 - 4^2)\,a_4 = -m^2(m^2 - 2^2)(m^2 - 4^2).$$

$$\cos(m \sin^{-1} x) = 1 - \frac{m^2}{2\,!}x^2 + \frac{m^2(m^2-2^2)}{4\,!}x^4$$
$$- \frac{m^2(m^2-2^2)(m^2-4^2)(m^2-6^2)}{6\,!}x^6 + \ldots$$

Here if we substitute $\sin\theta$ for x, we get

$$\cos m\theta = 1 - \frac{m^2}{2\,!}\sin^2\theta + \frac{m^2(m^2-2^2)}{4\,!}\sin^4\theta$$
$$- \frac{m^2(m^2-2^2)(m^2-4^2)(m^2-6^2)}{6\,!}\sin^6\theta + \ldots$$

Exercises XXXI.

1. Using Maclaurin's Theorem, expand the followi[ng] functions :—

(1) $\tan x$. (6) $e^{\sin x}$.
(2) $e^x \cos x$. (7) $e^x \sin x$.
(3) $\sec x$. (8) $\cos x \cosh x$.
(4) $e^x \sec x$. (9) $\sin x \sinh x$.
(5) $\log(1+e^x)$. (10) $\log \sec x$.

2. Prove the following expansions :—

(1) $x \cot x = 1 - \frac{x^2}{3} - \frac{x^4}{45} - \frac{2x^6}{945} \ldots$

(2) $\cos x \sinh x = x - \frac{1}{3}x^3 - \frac{1}{30}x^5 \ldots$

(3) $\tan\left(x + \frac{\pi}{4}\right) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \ldots$

(4) $\log(1 + \sin x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 \ldots$

(5) $\log(1 + x + x^2) = x + \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 \ldots$

3. Verify the following expansions of functions :—

(1) $\sin(x+a) = \sin a + x \cos a - \frac{x^2}{2!}\sin a - \frac{x^3}{3!}\cos a \ldots$

(2) $\tan(x+a) = \tan a + x \sec^2 a + x^2 \sec^2 a \tan a + \ldots$

(3) $\tan^{-1}(1+x) = \frac{\pi}{4} + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{12}x^3 \ldots$

4. Show that

$$\sin(m \sin^{-1} x) = mx - \frac{m(m^2-1^2)}{3!}x^3 + \frac{m(m^2-1^2)(m^2-3^2)}{5!}x^5 - \frac{m(m^2-1^2)(m^2-3^2)(m^2-5^2)}{7!}x^7 \ldots$$

5. Show that

$$(\sin^{-1} x)^2 = \frac{2x^2}{2!} + \frac{2^2}{4!}2x^4 + \frac{2^2 \cdot 4^2}{6!}2x^6 + \frac{2^2 \cdot 4^2 \cdot 6^2}{8!}2x^8 + \ldots$$

6. If $e^{a \sin^{-1} x} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$, show that

(1) $a_{n+2} = \frac{n^2 + a^2}{(n+1)(n+2)} a_n$.

(2) $e^{a \sin^{-1} x} = 1 + ax + \frac{a^2 x^2}{2!} + \frac{a(a^2+1)}{3!} x^3$
$$+ \frac{a^2(a^2+2^2)}{4!} x^4 + \ldots$$

7. If a_n is the coefficient of x^n in the expansion of $e^x \sin x$, show that

$$a_n - \frac{a_{n-1}}{1!} + \frac{a_{n-2}}{2!} - \frac{a_{n-3}}{3!} \ldots = \frac{\sin \frac{n\pi}{2}}{n!}.$$

8. Show that

$$e^{x \cos \alpha} \cos (x \sin \alpha) = 1 + x \cos \alpha + \frac{x^2}{2!} \cos 2\alpha + \frac{x^3}{3!} \cos 3\alpha + \ldots$$

9. Expand $\sin\left(\frac{\pi}{4} + \theta\right)$ in a series of powers of θ.

10. If $y = e^{\tan^{-1} x} = a_0 + a_1 x + a_2 x^2 + \ldots$, show that y satisfies the linear differential equation
$(1+x^2) y_{n+2} + \{2(n+1)x - 1\} y_{n+1} + n(n+1) y_n = 0$ and prove that $(n+2) a_{n+2} + n a_n = a_{n+1}$. (B.Sc. 51 T.U.)

11. Expand (*i*) $\frac{x}{e^x - 1}$, (*ii*) $\log_e (1 + \sin x)$ in ascending powers of x up to the term involving x^4. (B.E. 50 M.U.)

12. Use Maclaurin's theorem to prove that

$$e^{ax} \cos bx = 1 + rx \cos \theta + \ldots + \frac{r^n x^n \cos n\theta}{n!} + \ldots,$$

where $r^2 = a^2 + b^2$ and $\tan \theta = b/a$. (B.Sc. 53 Os.U.)

13. If $a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2}$ be three consecutive terms of the expansion of $(1 - x^2) \sin^{-1} x$ in powers of x, prove that $a_{n+2} = \frac{n-1}{n+2} a_n$; also that all even terms vanish, and that the expansion is $x - \frac{x^3}{3} - \frac{2}{3.5} x^5 - \frac{2.4}{3.5.7} x^7 \ldots$

14. If $y = y(x)$ is defined by $x = \sin \theta$, $y = \cosh 2\theta$ for $-\pi/2 < \theta < \pi/2$, prove that $y_{n+2} = (n^2 + 4) y_n$ $(n = 0, 1, 2 \ldots)$, where y_n is the derivative of y with respect to x at $x = 0$.

Hence expand y in ascending powers of x. [B.A. (Hons.) 59]

CHAPTER VIII

PARTIAL DIFFERENTIATION, ERRORS AND APPROXIMATIONS

§ 34·1. We have considered till now only functions of o variable but we come across functions involving more than o variable. For example, the area of a rectangle is a functio two variables, the length and the breadth of the rectangle.

If u be a function of two variables x and y, let us assu the functional relation as $u = f(x, y)$. Here x alone or y al or both x any y simultaneously may be varied and in each c a change in the value of u will result. Generally the chang the value of u will be different in each of these three cases. Si x and y are independent, x may be supposed to vary whe remains constant or the reverse.

The derivative of u with respect to x when x varies a y remains constant is called the partial derivative of u with resp to x and is denoted by the symbol $\frac{\partial u}{\partial x}$. We may then write

$$\frac{\partial u}{\partial x} = \underset{\Delta x \to 0}{\text{Lt}} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}.$$

Similarly, when x remains constant and y varies, the part derivative of u with respect to y is

$$\frac{\partial u}{\partial y} = \underset{\Delta x \to 0}{\text{Lt}} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.$$

$\frac{\partial u}{\partial x}$ is also written $\frac{\partial}{\partial x} f(x, y)$ or $\frac{\partial f}{\partial x}$.

Similarly, $\frac{\partial u}{\partial y}$ is also written $\frac{\partial}{\partial y} f(x, y)$ or $\frac{\partial f}{\partial y}$.

Successive partial derivatives.

Consider the function $u = f(x, y)$. Then in general $\frac{\partial u}{\partial x}$ and are functions of both x any y and may be differentiated ag with respect to either of the independent variables thus giving r to successive partial derivatives. Regarding x alone as vary we denote the results by

$$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}, \ldots\ldots \frac{\partial^n u}{\partial x^n}$$

or when y alone varies,

$$\frac{\partial^2 u}{\partial y^2}, \frac{\partial^3 u}{\partial y^3}, \ldots\ldots \frac{\partial^n u}{\partial y^n}.$$

If we differentiate u with respect to x regarding y as constant and then this result is differentiated with respect to y regarding x as constant, we obtain

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \text{ which we denote by } \frac{\partial^2 u}{\partial y\, \partial x}.$$

Similarly, if we differentiate u twice with respect to x and then once with respect to y, the result is denoted by the symbol $\frac{\partial^3 u}{\partial y\, \partial^2 x}$. The partial differential coefficient of $\frac{\partial u}{\partial y}$ with respect to x considering y as a constant is denoted by $\frac{\partial^2 u}{\partial x\, \partial y}$.

Generally, in the ordinary functions which we come across

$$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial^2 u}{\partial x\, \partial y}.$$

§ 34·2. **Function of function rule.** This rule is very useful in partial differentiation.

Let z be a function of u where u is a function of two independent variables x and y.

Then $\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y}$.

Let x and y receive arbitrary increments Δx and Δy and let the corresponding increments in u and z be Δu and Δz respectively.

Then $$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta u}\frac{\Delta u}{\Delta x}.$$

Proceeding to the limit when $\Delta x \to 0$,

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x}.$$

(Note that the straight 'd' is used in $\frac{dz}{du}$ as z is a function of only one variable u while the curved '∂' is used in $\frac{\partial u}{\partial x}$ as u is a function of two independent variables.)

Similarly the other result follows.

Examples.

Ex. 1. Find the partial differential coefficients of

$$u = \sin(ax + by + cz).$$

$$\frac{\partial u}{\partial x} = a\cos(ax + by + cz).$$

$$\frac{\partial u}{\partial y} = b\cos(ax + by + cz).$$

$$\frac{\partial u}{\partial z} = c\cos(ax + by + cz).$$

Ex. 2. If $u = \dfrac{xy}{x+y}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = u.$$

$$\frac{\partial u}{\partial x} = \frac{(x+y)\,y - xy}{(x+y)^2} = \frac{y^2}{(x+y)^2}.$$

Similarly $\dfrac{\partial u}{\partial y} = \dfrac{x^2}{(x+y)^2}.$

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{xy^2 + x^2y}{(x+y)^2} = \frac{xy}{x+y}$$
$$= u.$$

Ex. 3. If $u = \tan^{-1}\dfrac{x^3+y^3}{x-y}$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u. \quad \text{(B.Sc. 46 M ; B.A. 51 N}$$

$$\tan u = \frac{x^3+y^3}{x-y}.$$

Differentiating w.r.t. x alone,

$$\sec^2 u\,\frac{\partial u}{\partial x} = \frac{(x-y)\,3x^2 - (x^3+y^3)}{(x-y)^2}$$
$$= \frac{2x^3 - 3x^2y - y^3}{(x-y)^2}.$$

Similarly, $\sec^2 u\,\dfrac{\partial u}{\partial y} = \dfrac{x^3 + 3xy^2 - 2y^3}{(x-y)^2}.$

$$\therefore\quad \sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$
$$= \frac{x(2x^3 - 3x^2y - y^3) + y(x^3 + 3xy^2 - 2y^3)}{(x-y)^2}$$
$$= 2\frac{x^3+y^3}{x-y}$$
$$= 2\tan u.$$

$$\therefore\quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u\cos^2 u = \sin 2u.$$

Ex. 4. If $V = (x^2 + y^2 + z^2)^{-1/2}$, show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \qquad \text{(B.Sc. 46 M)}$$

Differentiating V with respect to x alone, we get

$$\frac{\partial V}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}\, 2x$$
$$= -x\,(x^2 + y^2 + z^2)^{-3/2}.$$

Differentiating once again with respect to x alone,

$$\frac{\partial^2 V}{\partial x^2} = \frac{3}{2}x\,(x^2 + y^2 + z^2)^{-5/2}\, 2x - (x^2 + y^2 + z^2)^{-3/2}$$
$$= \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}.$$

Similarly $\dfrac{\partial^2 V}{\partial y^2} = \dfrac{2y^2 - z^2 - x^2}{(x^2 + y^2 + z^2)^{5/2}}.$

$$\frac{\partial^2 V}{\partial z^2} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}.$$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

(V is said to satisfy Laplace's Equation.)

Ex. 5. Illustrate the theorem that

$$\frac{\partial^2 u}{\partial x\,.\,\partial y} = \frac{\partial^2 u}{\partial y\,.\,\partial x} \text{ when } u \text{ is equal to } \log\frac{x^2 + y^2}{xy}.$$

(B.A. 50 M)

$$u = \log\frac{x^2 + y^2}{xy} = \log(x^2 + y^2) - \log x - \log y.$$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2} - \frac{1}{x}$$

$$\frac{\partial^2 u}{\partial y\,.\,\partial x} = \frac{\partial}{\partial y}\left(\frac{2x}{x^2 + y^2} - \frac{1}{x}\right) = -\frac{4xy}{(x^2 + y^2)^2}.$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2} - \frac{1}{y}.$$

$$\frac{\partial^2 u}{\partial x\,.\,\partial y} = \frac{\partial}{\partial x}\left(\frac{2y}{x^2 + y^2} - \frac{1}{y}\right) = -\frac{4xy}{(x^2 + y^2)^2}.$$

$$\therefore \quad \frac{\partial^2 u}{\partial y\,.\,\partial x} = \frac{\partial^2 u}{\partial x\,.\,\partial y}.$$

Exercises XXXII.

1. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}.$$

2. If $u = \log(\tan x + \tan y + \tan z)$, show that
$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2.$$

3. If $u = (y - z)(z - x)(x - y)$, show that
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

4. If $u = \sin\left(\frac{x^2 + y^2}{x + y}\right)$, show that
$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{x^2 + y^2}{x + y} \cos\left(\frac{x^2 + y^2}{x + y}\right).$$

5. Given that $u = \tan^{-1}\left(\frac{x - y}{x + y}\right)^{3/2}$, prove that
$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0.$$

6. If $e^{-z/(x^2 - y^2)} = x - y$, prove that
$$y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = x^2 - y^2.$$

7. If $u = \sin^{-1} \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$, show that
$$\frac{\partial u}{\partial x} = -\frac{y}{x} \frac{\partial u}{\partial y}.$$

8. Verify that $\frac{\partial^2 u}{\partial x \,.\, \partial y} = \frac{\partial^2 u}{\partial y \,.\, \partial x}$ in the following cases:

(1) $u = \sin^{-1}\left(\frac{y}{x}\right)$. (B.A. 49 M)

(2) $u = x \sin y + y \sin x$. (B.A. 50 M)

(3) $u = x^y$. (B.Sc. 50 M)

(4) $u = \log\{x \tan^{-1}(x^2 + y^2)\}$.

9. If $u = \log(x^2 + y^2 + z^2)$, prove that
$$x \frac{\partial^2 u}{\partial y \,.\, \partial z} = y \frac{\partial^2 u}{\partial z \,.\, \partial x} = z \frac{\partial^2 u}{\partial x \,.\, \partial y}.$$

10. If $V = \log r$ where $r^2 = (x - a)^2 + (y - b)^2$ and $(x - a)$ and $(y - b)$ are not simultaneously zero, prove that
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$
(B.A. 49 M ; B.A. 52 M)

11. If $u = \dfrac{1}{r}$ and $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$,

prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0.$ (B.Sc. 56 M)

12. If $f(x, y) = \log \sqrt{x^2 + y^2}$, find the value of

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$ (B.A. Sub. 52)

13. If $u = \sin^{-1} \dfrac{x+y}{\sqrt{x} + \sqrt{y}}$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u.$$ (B.Sc. 52 M)

14. If $z = 3xy - y^3 + (y^2 - 2x)^{3/2}$, show that

$$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x \cdot \partial y}\right)^2.$$ (B.A. Sub. 50)

15. If $u = \log(x^2 + y^2 + z^2)$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{x^2 + y^2 + z^2}.$$

16. If $z = \tan(y + ax) + (y - ax)^{3/2}$, find the value of

$$\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2}.$$ (B.A. Sub. 52)

17. If $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$, prove that

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}.$$

18. If $V = e^{a\theta}\cos(a \log r)$, show that

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0.$$

19. If $r^2 = x^2 + y^2$, prove that

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right\}.$$

20. If $u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

21. If $u = e^{xy}$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{u}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right].$$

22. If $u = \sin^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$$ (B.A. 57)

23. If $u = \tan^{-1}\left(\dfrac{y}{x}\right)$, verify that

(a) $\dfrac{\partial^3 u}{\partial y^2 \cdot \partial x} = \dfrac{\partial^3 u}{\partial x \cdot \partial y^2} = \dfrac{\partial^3 u}{\partial y \cdot \partial x \cdot \partial y}$.

(b) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$ (B.Sc. 53 Os.U

24. If $x^x y^y z^z = c$, show that when $x = y = z$,

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = -(x \log ex)^{-1}.$$ (B.Sc. 51 T.U

25. If $z = x \tan^{-1}\dfrac{y}{x} + xe^{x/y}$, show that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2\,xy \frac{\partial^2 z}{\partial x \cdot \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0.$$

26. If $V = \dfrac{xz}{x^2 + y^2}$, prove that V satisfies

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

27. If $z = f(x^2 + y^2)$, show that $x\dfrac{\partial z}{\partial y} - y\dfrac{\partial z}{\partial x} = 0.$

(B.Sc. 56

28. If $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\frac{\partial^2 \theta}{\partial x\, \partial y} = \frac{\partial^2 (\log r)}{\partial x^2} = -\frac{\partial^2 (\log r)}{\partial y^2} = -\frac{1}{r^2}\cos 2\theta.$$

(B.Sc. 55

29. If $u = x^2 \tan^{-1}\dfrac{y}{x} - y^2 \tan^{-1}\dfrac{x}{y}$, prove that

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{x^2 - y^2}{x^2 + y^2}.$$ (B.A. 5

30. If $z = e^x (x\cos y - y \sin y)$, show that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$

31. If $x = e^{-t}\cos\theta$, $y = e^{-t}\sin\theta$, prove that

$$\frac{\partial t}{\partial x} = -\frac{x}{x^2 + y^2} \text{ and } \frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2}.$$

(B.Sc. 65

32. If $z = \sin(x - y) + \log(x + y)$, show that $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial y^2}$

(B.Sc. 65

§ 34·3. Total differential coefficient.

If u be a continuous function of x and y and if x and y recei small increments Δx and Δy (which are usually quite independe of one another), u will receive in turn a small increment Δ Then

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y).$$

This quantity Δu is called the total increment of u.

$$\Delta u = f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) + f(x, y+\Delta y) - f(x, y).$$

Applying the theorem of mean value to each of the two differences on the right-hand side,

$$f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) = f_x'(x+\theta_1 \Delta x, y+\Delta y)\,\Delta x\,;$$

$$f(x, y+\Delta y) - f(x, y) = f_y'(x, y+\theta_2 \Delta y)\,\Delta y,$$

where f_x' and f_y' denote the partial differential coefficients with respect to x and y respectively and where θ_1 and θ_2 are positive fractions.

$$\therefore \quad \Delta u = f_x'(x+\theta_1 \Delta x, y+\Delta y)\,\Delta x + f_y'(x, y+\theta_2 \Delta y)\,\Delta y.$$

If x and y and therefore also u are continuous functions of some other variable t and if Δx, Δy and Δu be the increments of x, y and u due to an increment Δt of t, dividing Δu by Δt, we get

$$\frac{\Delta u}{\Delta t} = f_x'(x+\theta_1 \Delta x, y+\Delta y)\frac{\Delta x}{\Delta t} + f_y'(x, y+\theta_2 \Delta y)\frac{\Delta y}{\Delta t}.$$

Now let $\Delta t \to 0$.

Then $\dfrac{du}{dt} = f_x'(x, y)\dfrac{dx}{dt} + f_y'(x, y)\dfrac{dy}{dt}$.

$$\therefore \quad \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\,\frac{dy}{dt}.$$

In the differential form, this can be written as

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

du is called the total differential of u.

In the same way, if $u = f(x, y, z)$ and x, y, z are all functions of t, we get

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}.$$

And similarly if $u = f(x_1, x_2, \ldots\ldots x_n)$ where $x_1, x_2, \ldots\ldots x_n$ are known functions of a variable t, we have the relation

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\cdot\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}\cdot\frac{dx_2}{dt} + \ldots\ldots + \frac{\partial u}{\partial x_n}\cdot\frac{dx_n}{dt}$$

$$\text{or } du = \frac{\partial x}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 + \ldots\ldots + \frac{\partial u}{\partial x_n}dx_n.$$

§ 34·4. A special case.

If $u = f(x, y)$ where x and y are functions of t, we get

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}.$$

If we take t to be x, we get u as a function of x and y, where y is a function of x.

Since $\frac{dx}{dt}$ is now unity, this relation becomes

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}.$$

The quantities $\frac{du}{dx}$ and $\frac{\partial u}{\partial x}$ are quite distinct. For example let $u = x^2 + 2xy + y^2$ and let y be a function of x.

Then $\frac{\partial u}{\partial x} = 2x + 2y$; $\frac{\partial u}{\partial y} = 2x + 2y$.

$$\therefore \quad \frac{du}{dx} = 2x + 2y + (2x + 2y)\frac{dy}{dx}$$

and the value of $\frac{dy}{dx}$ will depend on the relation between x and y.

§ 34·5. Implicit functions.

If the relation between x and y be given in the form $f(x, y) = c$ where c is a constant, then the total differential coefficient with respect to x is zero, since the differential coefficient of a constant is zero ; hence

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}.$$

This gives an alternative method of finding the differential coefficient of y with respect to x when y is given as an implicit function of x.

Examples.

Ex. 1. Find $\frac{du}{dt}$ where $u = x^2 + y^2 + z^2$, $x = e^t$, $y = e^t \sin t$ and $z = e^t \cos t$.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$
$$= 2xe^t + 2y\,(e^t \sin t + e^t \cos t) + 2z\,(e^t \cos t - e^t \sin t)$$
$$= 2e^t\,(x + y \sin t + y \cos t + z \cos t - z \sin t)$$
$$= 2e^t\,(e^t + e^t \sin^2 t + e^t \sin t \cos t + e^t \cos^2 t - e^t \sin t \cos t)$$
$$= 2e^t \,.\, 2e^t$$
$$= 4e^{2t}.$$

Ex. 2. Find $\dfrac{du}{dx}$ when $u = x^2 + y^2$, where

$$y = \frac{1-x}{x}.$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$
$$= 2x + 2y\frac{d}{dx}\left(\frac{1-x}{x}\right)$$
$$= 2x - \frac{2y}{x^2}$$
$$= 2x - \frac{2(1-x)}{x^3}$$
$$= \frac{2(x^4 + x - 1)}{x^3}.$$

Ex. 3. If $x^3 + y^3 = 3axy$, find $\dfrac{dy}{dx}$.

$$x^3 + y^3 - 3axy = 0, \quad i.e., \quad f(x, y) = 0.$$
$$\frac{\partial f}{\partial x} = 3x^2 - 3ay$$
$$\frac{\partial f}{\partial y} = 3y^2 - 3ax.$$
$$\therefore \quad \frac{dy}{dx} = -\frac{3x^2 - 3ay}{3y^2 - 3ax}$$
$$= -\frac{x^2 - ay}{y^2 - ax}.$$

§ 34·5. Homogeneous functions.

Let us consider the function

$$f(x, y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_{n-1}xy^{n-1} + a_ny^n.$$

In this expression the sum of the indices of the variables x and y in each term is n. Such an expression is called a homogeneous

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function of degree n. This expression can be written as follows

$$f(x, y) = x^n\left(a_0 + a_1\frac{y}{x} + a_2\frac{y^2}{x^2} + \dots + a_n\frac{y^n}{x^n}\right)$$

$$= x^n \times \text{a function of } \frac{y}{x}$$

$$= x^n F\left(\frac{y}{x}\right).$$

Similarly, a homogeneous function of degree n consisting of m variables $x_1, x_2, \dots, x_m$ can be written as

$$x_r^n F\left(\frac{x_1}{x_r}, \frac{x_2}{x_r}, \dots, \frac{x_m}{x_r}\right).$$

Euler's Theorem

If $f(x, y)$ is a homogeneous function of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$$

This is known as *Euler's Theorem* on homogeneous functions.

Let us assume that

$$f(x, y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_n y^n$$

$$= x^n F\left(\frac{y}{x}\right).$$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left\{x^n F\left(\frac{y}{x}\right)\right\} = nx^{n-1}F\left(\frac{y}{x}\right) - x^n F'\left(\frac{y}{x}\right)\cdot\frac{y}{x^2}$$

$$= nx^{n-1}F\left(\frac{y}{x}\right) - x^{n-2}yF'\left(\frac{y}{x}\right).$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left\{x^n F\left(\frac{y}{x}\right)\right\} = x^n F'\left(\frac{y}{x}\right)\cdot\frac{1}{x} = x^{n-1}F'\left(\frac{y}{x}\right)$$

$$\therefore x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n F\left(\frac{y}{x}\right) - x^{n-1}yF'\left(\frac{y}{x}\right) + x^{n-1}yF'\left(\frac{y}{x}\right)$$

$$= nx^n F\left(\frac{y}{x}\right)$$

$$= nf.$$

In general if $f(x_1, x_2, \dots, x_m)$ is a homogeneous function of degree n, then

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \dots + x_m\frac{\partial f}{\partial x_m} = nf.$$

Example.

Verify Euler's Theorem when $u = x^3 + y^3 + z^3 + 3xyz$.

$$\frac{\partial u}{\partial x} = 3x^2 + 3yz.$$

$$\frac{\partial u}{\partial y} = 3y^2 + 3zx.$$

$$\frac{\partial u}{\partial z} = 3z^2 + 3xy.$$

$$\begin{aligned}\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} &= x\,(3x^2 + 3yz) + y\,(3y^2 + 3zx) \\ &\qquad + z\,(3z^2 + 3xy) \\ &= 3\,(x^3 + y^3 + z^3 + 3xyz) \\ &= 3u.\end{aligned}$$

§ 34·6. Partial derivatives of a function of two functions.

Let $V = F\ (u, v)$ where $u = f(x, y)$, $v = f_1(x, y)$ and x, y are independent variables.

If we write V in the form $F\{f(x, y), f_1(x, y)\}$, we can obtain $\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}$ by the ordinary rules of partial differentiation but it is usually done without substitution.

By definition since x, y are independent,

$$dV = \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy. \tag{1}$$

u is a function of x and y.

$$\therefore\ du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy. \tag{2}$$

v is a function of x and y.

$$\therefore\ dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy. \tag{3}$$

V is a function of u and v.

$$\therefore\ dV = \frac{\partial V}{\partial u}\,du + \frac{\partial V}{\partial v}\,dv. \tag{4}$$

Substituting the values of du and dv from (2) and (3) in (4), we get

$$\begin{aligned}dV &= \frac{\partial V}{\partial u}\left(\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy\right) + \frac{\partial V}{\partial v}\left(\frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy\right) \\ &= \left(\frac{\partial V}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial V}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial y}\right)dy. \qquad (5)\end{aligned}$$

Comparing (1) and (5), we get

$$\frac{\partial V}{\partial x}=\frac{\partial V}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial V}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial V}{\partial y}=\frac{\partial V}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial V}{\partial y}\cdot\frac{\partial v}{\partial y}.$$

These results may be expressed by saying that the operat $\frac{\partial}{\partial x}$ and $\left(\frac{\partial u}{\partial x}\cdot\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}\cdot\frac{\partial}{\partial v}\right)$ are equivalent.

Similarly $\frac{\partial}{\partial y}\equiv\frac{\partial u}{\partial y}\cdot\frac{\partial}{\partial u}+\frac{\partial v}{\partial y}\cdot\frac{\partial}{\partial v}.$

$$\frac{\partial^2 V}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)$$

$$=\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial V}{\partial x}\right).$$

$$\frac{\partial^2 V}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right)$$

$$=\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u}+\frac{\partial v}{\partial y}\cdot\frac{\partial}{\partial v}\right)\frac{\partial V}{\partial y}.$$

In this way, it is possible to express higher par derivatives.

Examples.

Ex. 1. If $z=f(x, y)$ and $x=r\cos\theta$, $y=r$ sin prove that $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$. (B.Sc. 51

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}.$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}.$$

$$x=r\cos\theta.$$

$$\therefore \frac{\partial x}{\partial r}=\cos\theta\ ;\quad \frac{\partial x}{\partial \theta}=-r\sin\theta.$$

$$y=r\sin\theta.$$

$$\therefore \frac{\partial y}{\partial r}=\sin\theta\ ;\quad \frac{\partial y}{\partial \theta}=r\cos\theta.$$

Hence $\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$

$$=\cos\theta\frac{\partial z}{\partial x}+\sin\theta\frac{\partial z}{\partial y}$$

$$\text{and } \frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$=-r\sin\theta\frac{\partial z}{\partial x}+r\cos\theta\frac{\partial z}{\partial y}.$$

$$\therefore \left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2.$$

$$\therefore \left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

Ex. 2. Transform $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}$ into polar coordinates. (B.Sc. 45 M)

We have $x=r\cos\theta$, $y=r\sin\theta$

and $r^2=x^2+y^2$, $\tan\theta=\frac{y}{x}$.

$$\therefore 2r\frac{\partial r}{\partial x}=2x. \quad \therefore \frac{\partial r}{\partial x}=\frac{x}{r}=\cos\theta$$

$$\text{and } \sec^2\theta\,\frac{\partial \theta}{\partial x}=-\frac{y}{x^2}. \quad \therefore \frac{\partial \theta}{\partial x}=-\frac{y}{r^2}=-\frac{\sin\theta}{r}.$$

$$\therefore \frac{\partial V}{\partial x}=\frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial V}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}=\cos\theta\frac{\partial V}{\partial r}-\frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}.$$

Thus $\frac{\partial}{\partial x}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$.

$$\therefore \frac{\partial^2 V}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)$$

$$=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial V}{\partial r}-\frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}\right)$$

$$=\cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial V}{\partial r}-\frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial V}{\partial r}-\frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}\right)$$

$$=\cos\theta\left\{\cos\theta\frac{\partial^2 V}{\partial r^2}-\frac{\sin\theta}{r}\frac{\partial^2 V}{\partial r\,\partial \theta}+\frac{\sin\theta}{r^2}\frac{\partial V}{\partial \theta}\right\}$$

$$-\frac{\sin\theta}{r}\left\{-\sin\theta\frac{\partial V}{\partial r}+\cos\theta\frac{\partial^2 V}{\partial \theta\,\partial r}-\frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}-\frac{\sin\theta}{r}\cdot\frac{\partial^2 V}{\partial \theta^2}\right\}.$$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} = \cos^2\theta \frac{\partial^2 V}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\,\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial\theta^2}$$
$$+ \frac{\sin^2\theta}{r}\frac{\partial V}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial\theta}$$

assuming that $\dfrac{\partial^2 V}{\partial r\,\partial\theta} = \dfrac{\partial^2 V}{\partial\theta\,\partial r}$.

To get $\dfrac{\partial}{\partial y}$, we note that we change θ in $\dfrac{\partial}{\partial x}$ to $\dfrac{\pi}{2} - \theta$.

Hence $\dfrac{\partial}{\partial y} = \sin\theta \dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial}{\partial\theta}$.

Similarly, $\dfrac{\partial^2 V}{\partial y^2}$ can be found from $\dfrac{\partial^2 V}{\partial x^2}$ by replacing θ by $\dfrac{\pi}{2} - \theta$. This gives

$$\frac{\partial^2 V}{\partial y^2} = \sin^2\theta\frac{\partial^2 V}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\,\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial\theta^2}$$
$$+ \frac{\cos^2\theta}{r}\frac{\partial V}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial\theta}$$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{1}{r}\frac{\partial V}{\partial r}.$$

Exercises XXXIII.

1. Find $\dfrac{du}{dt}$ in the following cases :—

(1) $u = \log(x + y + z)$ where $x = \cos t$, $y = \sin^2 t$, $z = \cos t$.

(2) $u = xyz$ where $x = e^{-t}$, $y = e^{-t}\sin^2 t$, $z = \sin t$.

(3) $u = x^3y^4z^2$ where $x = t^2$, $y = t^3$, $z = t^4$.

(4) If $u = \sin(xy^2)$, $x = \log t$, $y = e^t$. (B.Sc. 47)

2. Find $\dfrac{du}{dx}$ in the following cases :—

(1) $u = x^2 + y^2 + a^2$ where $x^3 + y^3 = a^3$.

(2) $u = \tan^{-1}\left(\dfrac{y}{x}\right)$ where $y = \tan^2 x$.

(3) $u = x^3y^2$ where $x^2 - xy + y^2 = a^2$.

(4) $u = x\log(xy)$ where $x^3 + y^3 - 3axy = 0$.

(5) $u = \sin(x^2 + y^2)$ where $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

3. Find $\frac{dy}{dx}$ if x and y are related as follows :—

(a) $y^2 = 4ax$.

(b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(c) $xy = c^2$.

(d) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

(e) $(x - a)^2 + (y - b)^2 = r^2$.

(f) $y = b \tan^{-1} \left\{ \frac{x}{a} + \tan^{-1} \frac{y}{x} \right\}$. (B.Sc. 51 M)

4. Verify Euler's theorem on Homogeneous functions in the case of the following functions :—

(1) $u = x^3 - 2x^2y + 3xy^2 + y^3$.

(2) $u = \frac{x^2 (x^2 - y^2)^3}{(x^2 + y^2)^3}$.

(3) $u = \sin \left(\frac{x - y}{x + y} \right)^{1/2}$.

5. If $u = f(x, y) = F(r, \theta)$ where $x = r \cos \theta$, $y = r \sin \theta$, prove that

$$r \frac{\partial u}{\partial r} = x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \text{ and } \frac{\partial u}{\partial \theta} = -y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y}.$$

(B.Sc. 52 M)

6. If z be a function of x and y and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, then prove that

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}.$$ (B.A. 52 M)

7. (1) If $u = f(x - y, y - z, z - x)$, show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$ (B.Sc. 45 M)

(2) If $u = (x - y)^4 + (y - z)^4 + (z - x)^4$, show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$ (B.Sc. 47 M)

8. If $V = f\left(\frac{x}{z}, \frac{y}{z}\right)$, prove that

$$x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial z} = 0.$$

9. If $V = f(x, y)$ and if $x = u^2 - v^2$, $y = 2uv$, prove that

(1) $u \frac{\partial V}{\partial u} - y \frac{\partial V}{\partial y} = 2 (u^2 + v^2) \frac{\partial V}{\partial x}$;

$$(2)\ \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\frac{1}{4\,(u^2+v^2)}\left(\frac{\partial^2 V}{\partial u^2}+\frac{\partial^2 V}{\partial v^2}\right).$$

10. If z is a function of x and y and if $x=u-v$, $y=$ ... prove that

$$(1)\ (u+v)\frac{\partial z}{\partial x}=u\frac{\partial z}{\partial u}-v\frac{\partial z}{\partial v}\ ;$$

$$(2)\ (u+v)\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}.$$

11. If z is a function of x and y and if $x=e^u \sin v$, $y=e^u$ cos ... prove that

$$(1)\ \frac{\partial z}{\partial u}=x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\ ;$$

$$(2)\ \frac{\partial z}{\partial x}=e^{-u}\left(\sin v\,\frac{\partial z}{\partial u}+\cos v\,\frac{\partial z}{\partial v}\right).$$

12. If $x=X\cos\alpha-Y\sin\alpha$, $y=X\sin\alpha+Y\cos\alpha$; pro... that if u is any function of x and y,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial X^2}+\frac{\partial^2 u}{\partial Y^2}.$$ (B.Sc. 52 T.U...)

13. If $y=f(x,y)$; $z=\phi\,(x,y)$, evaluate $\frac{dy}{dx}$ in terms of t... partial derivatives of f and ϕ.

14. If $u=f(t)$ and $v=\phi\,(t)$ where $t=f\,(x,y)$, prove th...

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}=\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$$ (B.Sc. 52 T.U...)

15. If $V=(1-2xy+y^2)^{1/2}$, show that

$$x\frac{\partial V}{\partial x}-y\frac{\partial V}{\partial y}=y^2V^3 \text{ and}$$

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial V}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{y^2\frac{\partial V}{\partial y}\right\}=0.$$

(B.Sc. 51 T.U...)

16. If $u=x\phi\left(\frac{y}{x}\right)+\Psi\left(\frac{y}{x}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$$ (B.Sc. 52 T.U...)

17. If $u^2(x^2+y^2+z^2)=1$, show that

$$\Sigma\frac{\partial^2 u}{\partial x^2}=0 \text{ and } u+\Sigma\, x\frac{\partial u}{\partial x}=0.$$ (B.Sc. ...)

ERRORS AND APPROXIMATIONS

§ 35·1. Small corrections.

In practice, all measurements are subject to errors and it is therefore of importance to find the total effect of small errors in the observed values of the several variables, on a quantity which depends on these variables.

Let a quantity x be determined by measurements and y be a function of x, *i.e.*, $y = f(x)$. Suppose the value of x given by the measurement differs from its true value by Δx; then the true value of y is $f(x + \Delta x)$ and the error Δy is

$$\Delta y = f(x + \Delta x) - f(x)$$

$$= f'(x + \theta \Delta x)\, \Delta x \text{ where } 0 < \theta < 1$$

(By Mean Value Theorem)

$$= f'(x)\, \Delta x \text{ approximately} \quad [\text{as } f'(x) \text{ is continuous}]$$

neglecting $\theta \Delta x$ as very small, since Δx is small and θ is a positive fraction.

This result can be derived by the following method also.

If $y = f(x)$ and if Δx, Δy be simultaneous increments in x and y, the limiting value of the ratio $\dfrac{\Delta y}{\Delta x}$, when Δx is indefinitely diminished is, by definition, $f'(x)$.

Hence $\dfrac{\Delta y}{\Delta x} = f'(x) + \epsilon$ where ϵ is an ultimately vanishing quantity.

$$\therefore \quad \Delta y = f'(x)\, \Delta x + \epsilon\, \Delta x.$$

As Δx approaches the value zero, the second term on the right-hand side becomes more and more insignificant compared with the first, since the limiting value of ϵ is zero. Hence it becomes more and more nearly true that

$$\Delta y = f'(x)\, \Delta x.$$

Δx is called the absolute error in x; the relative error in x is $\dfrac{\Delta x}{x}$ and percentage error is $\dfrac{\Delta x}{x} \cdot 100$. Thus if an error of $\cdot 1''$ is committed in measuring a length of $5''$, the absolute error is $\cdot 1''$, the relative error is $\cdot 02$ and the percentage error is 2.

§ 35·2. Increments and differentials.

So far we have spoken of the differential coefficient $\dfrac{dy}{dx}$ and in that notation, we have insisted upon the fact that $\dfrac{dy}{dx}$ is an operator

and that dx and dy are not separate entities. But, for convenien[ce] it is often necessary to assign separate meanings to dx and [dy]. Thus we are led to the notion of *differentials*.

Definition. dx and dy are said to be differentials in x and [y] if they be *any* two small quantities such that the ratio of dy to [dx] is the derivative $\frac{dy}{dx}$.

It must be emphasised in this connection that when we spe[ak] of the derivative $\frac{dy}{dx}$, the idea of quotient is absent and that t[he] derivative represents a certain limit process.

In other words, the differential relation is

$$dy = dx. \left(\text{derivative } \frac{dy}{dx}\right) = dx \,.\, f'(x) \qquad (1)$$

where $y = f(x)$.

The differentials dx and dy are to be distinguished fr[om] the increments Δx and Δy. In fact, we have seen above th[at]

$$\Delta y = f'(x)\, \Delta x$$

is only an approximate relation while (1) represents an ex[act] relation.

The distinction between differentials and increments c[an] best be illustrated graphically.

Let P be any point (x, y) on the curve $y = f(x)$ and l[et] Q be $(x + \Delta x, y + \Delta y)$ on the curve (*vide* Fig. 21). Draw P[M,] QN the ordinates at P and Q and draw PR parallel to Ox. Th[en]

$MN = \Delta x = PR$ and $\Delta y = RQ$; tan $\angle PQR = \frac{\Delta y}{\Delta x}$.

When $Q \to P$ along the curve, the chord QP becomes t[he] tangent PL meeting NQ at L.

$\angle QPR \to \Psi$, the inclination of the tangent to Ox.

$$\text{Tan } \Psi = \underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

In ΔLPR, $\angle LPR = \Psi$.

$$\therefore\ LR = PR \tan \Psi = \Delta x \frac{dy}{dx} = \Delta x\ f'(x).$$

Let dx and dy be differentials in x and y so that $dy = f'(x)$ (by definition).

If we choose to set $dx = \Delta x$, then dy is given by $f'(x) \Delta x = LR$.

But Δy, the increment in y, is RQ.

$\therefore$ $dy - \Delta y = RQ - RL = LQ$ which tends to zero as Q approaches P.

Examples.

Ex. 1. The time T of a complete oscillation of a simple pendulum of length l is given by the equation $T = 2\pi\sqrt{\frac{l}{g}}$ where g is a constant. Find the approximate error in the calculated value of T corresponding to an error of 2 per cent. in the value of l. (B.A. Sub. 37)

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Equating the differential on both sides after taking logarithms on both sides, we get

$$\log T = \log(2\pi) + \tfrac{1}{2}\log(l) - \tfrac{1}{2}\log(g). \qquad (1)$$

$$\therefore \frac{dT}{T} = \frac{1}{2}\frac{dl}{l}.$$

$\therefore$ The error relation is $\dfrac{\Delta T}{T} = \dfrac{1}{2}\dfrac{\Delta l}{l}$ (approximately).

Error in the value of l is 2 per cent.

$$\therefore \quad \frac{\Delta l}{l} \times 100 = 2.$$

$$\therefore \quad \frac{\Delta T}{T} \times 100 = \frac{1}{2}\frac{\Delta l}{l} \times 100 = \frac{1}{2} \times 2 = 1.$$

The percentage error in the calculated value of T is 1.

Ex. 2. $ABCD$ is a regular protractor in which $AB = 6$ inches, $BC = 2$ inches and O is the mid-point of AB. An angle BOP is indicated by a mark P on the edge CD. If in setting an angle θ degrees, a mark is made $\dfrac{1}{100}$ of an inch along the edge from the correct spot, show that the error in the angle is $\dfrac{9\sin^2\theta}{10\pi}$ degrees approximately. (B.A. Sub. 46)

Let PC be x ins. From P draw PQ perpendicular to AB.
Then $OQ = 3 - x$.

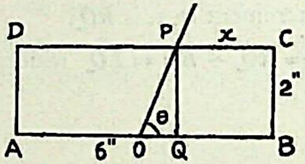

Fig. 16

$$\therefore \quad \tan\theta = \frac{PQ}{OQ} = \frac{2}{3-x}.$$

Taking differentials on both sides,

$$\therefore \quad \sec^2\theta\, d\theta = \frac{2}{(3-x)^2}\,dx$$

but $\quad (3-x) = 2\cot\theta.$

$$\therefore \quad \sec^2\theta\, d\theta = \frac{2}{4\cot^2\theta}\,dx.$$

$$\therefore \quad d\theta = \frac{\sin^2\theta}{2}\,dx.$$

Replacing the differentials dx and $d\theta$ by the increments Δx and $\Delta\theta$ respectively, we have the approximate error relation,

i.e., $\Delta\theta = \dfrac{\sin^2\theta}{2}\,\Delta x.$

$\Delta x = \dfrac{1}{100}$ and $\Delta\theta$ is measured in radians.

$$\therefore \quad \Delta\theta = \frac{\sin^2\theta}{2} \times \frac{1}{100} \times \frac{180}{\pi} \text{ degrees}$$

$$= \frac{9\sin^2\theta}{10\pi} \text{ degrees.}$$

Ex. 3. The angle C of a triangle ABC is found by actual measurement to be 47 degrees and the area is calculated by the formula $\frac{1}{2}ab\sin C$. Find the percentage error in the calculated area due to an error of $10'$ in the measured value of C.

Let S be the area of the triangle.

Then $S = \frac{1}{2}ab\sin C.$

Taking logarithms on both sides, we get

$$\log S = -\log 2 + \log a + \log b + \log\sin C.$$

Equating the differentials on both sides, we get

$$\frac{dS}{S} = \cot C\, dC$$

since there is no error in the measurements of a and b.

The error relation is $\frac{\Delta S}{S} = \cot C\, \Delta C$ approximately.

Percentage error in $S = \frac{\Delta S}{S} 100$

$$= \cot C \times \Delta C \times 100.$$

It is given that $\Delta C = 10' = \frac{1}{6} \times \frac{\pi}{180}$ radians.

$$\therefore \quad \text{Percentage error} = \frac{\cot 47° \times \pi}{6 \times 180} \times 100$$

$$= \frac{5\pi}{54} \cot 47°.$$

§ 35·3. Approximations in the case of two or more variables.

Let u be a function of x and y.

$$u = f(x, y).$$

If x and y are functions of t, we get

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}.$$

Hence this theorem may be written in the form

$$\Delta u = \left(\frac{\partial u}{\partial x} + \epsilon\right) \Delta x + \left(\frac{\partial u}{\partial y} + \eta\right) \Delta y,$$

where ϵ and η are very small.

This result can be expressed as

$$\Delta u = \frac{\partial u}{\partial x} \cdot \Delta x + \frac{\partial u}{\partial y} \cdot \Delta y \text{ approximately}$$

since $\epsilon\ \Delta x$ and $\eta\ \Delta y$ are insignificant compared with the other terms and so these terms can be discarded.

Examples.

Ex. 1. If the radius r of a sphere is found by weighing it in air and in water, prove that the relative error due to small errors in these weighings is $\frac{\Delta r}{r} = \frac{\Delta W_1 - \Delta W_2}{3 (W_1 - W_2)}$ where W_1 and W_2 are weights in air and water respectively.

$$W_1 - W_2 = \text{Weight of water displaced}$$

$$= \frac{4}{3}\pi r^3 \times \text{wt. per unit volume of water}$$

$$= \frac{4}{3}\pi r^3 \sigma, \text{ where } \sigma \text{ is the weight per unit volume.}$$

$$\therefore \quad \log (W_1 - W_2) = \log\left(\frac{4}{3}\pi\sigma\right) + 3 \log r.$$

Taking differentials on both sides, we get

$$3\frac{dr}{r}=\frac{d}{dW_1}\log(W_1-W_2)\,dW_1+\frac{d}{dW_2}\log(W_1-W_2)\,dW_2$$

$$=\frac{dW_1}{W_1-W_2}-\frac{dW_2}{W_1-W_2}$$

$$=\frac{dW_1-dW_2}{W_1-W_2}.$$

$\therefore$ The error relation is $\dfrac{\Delta r}{r}=\dfrac{\Delta W_1-\Delta W_2}{3(W_1-W_2)}$.

Ex. 2. The torsional rigidity of a length of a wire is obtained from the formula $N=\dfrac{8\pi Il}{t^2r^4}$. If l is decreased by 2 %, r increased by 2 %, t increased by 1·5 %, show that the value of N is diminished by 13 % approximately. (B.A. Sub. ...)

l is decreased by 2 %. $\therefore\ \dfrac{\Delta l}{l}\times 100=-2.$

r is increased by 2 %. $\therefore\ \dfrac{\Delta r}{r}\times 100=2.$

t is increased by 1·5 %. $\therefore\ \dfrac{\Delta t}{t}\times 100=\dfrac{3}{2}.$

$$N=\frac{8\pi Il}{t^2r^4}.$$

Taking logarithms on both sides, we get

$$\log N=\log(8\pi I)+\log l-2\log t-4\log r.$$

Taking differentials on both sides, we get

$$\frac{dN}{N}=\frac{dl}{l}-\frac{2dt}{t}-4\frac{dr}{r}.$$

The approximate error relation is got by replacing each of the differentials by the corresponding increment.

$$\therefore\ \frac{\Delta N}{N}\times 100=\frac{\Delta l}{l}100-\frac{2\Delta t}{t}100-\frac{4\Delta r}{r}100$$

$$=-2-3-8$$

$$=-13.$$

$\therefore$ The value of N is diminished by 13 %.

Ex. 3. The area of a triangle ABC is determined from the side a and the two angles B and C. If there are small errors

the values of B and C, show that the resulting error in the calculated value of the area will be $\frac{1}{2}(b^2 \Delta C + c^2 \Delta B)$.

(B.A. 38 M)

$$S = \frac{1}{2} ab \sin C \text{ and } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$\therefore \quad S = \frac{1}{2} a^2 \frac{\sin B \cdot \sin C}{\sin A}.$$

$$\therefore \quad \log S = \log\left(\tfrac{1}{2} a^2\right) + \log \sin B + \log \sin C - \log \sin A.$$

Equating the differentials on both sides,

$$\frac{dS}{S} = \frac{\cos B}{\sin B} dB + \frac{\cos C}{\sin C} dC - \frac{\cos A}{\sin A} dA.$$

Since $A + B + C = 180°$, $\therefore \quad dA + dB + dC = 0$.

$$\therefore \quad \frac{dS}{S} = \frac{\cos B}{\sin B} dB + \frac{\cos C}{\sin C} dC + \frac{\cos A}{\sin A}(dB + dC)$$

$$= \left(\frac{\cos B}{\sin B} + \frac{\cos A}{\sin A}\right) dB + \left(\frac{\cos C}{\sin C} + \frac{\cos A}{\sin A}\right) dC$$

$$= \frac{\sin(A+B)}{\sin A \sin B} dB + \frac{\sin(A+C)}{\sin A \sin C} dC$$

$$= \frac{\sin C}{\sin A \sin B} dB + \frac{\sin B}{\sin A \sin C} dC.$$

Since $S = \frac{1}{2} b^2 \frac{\sin A \sin C}{\sin B} = \frac{1}{2} c^2 \frac{\sin A \sin B}{\sin C}$

$$\frac{dS}{S} = \frac{1}{2} \frac{c^2\, dB}{S} + \frac{1}{2} \frac{b^2\, dC}{S}, \quad \text{i.e.,} \quad dS = \tfrac{1}{2}(c^2\, dB + b^2\, dC).$$

$\therefore$ Error is given by $\Delta S = \frac{1}{2}(c^2 \Delta B + b^2 \Delta C)$.

Ex. 4. Express S the area of a triangle ABC as a function of a, b, c and establish the formula

$$dS = R(\cos A\, da + \cos B\, db + \cos C\, dc).$$

$$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

$$\therefore \log S = \tfrac{1}{2} \log s + \tfrac{1}{2} \log(s-a) + \tfrac{1}{2} \log(s-b) + \tfrac{1}{2} \log(s-c)$$

$$= \tfrac{1}{2} \log(a+b+c) + \tfrac{1}{2} \log(b+c-a)$$

$$+ \tfrac{1}{2} \log(c+a-b) + \tfrac{1}{2} \log(a+b-c) - 2 \log 2.$$

Taking differentials on both sides,

$$\frac{dS}{S} = \Sigma \frac{1}{2}\left(\frac{1}{a+b+c} - \frac{1}{b+c-a} + \frac{1}{c+a-b} + \frac{1}{a+b-c}\right) da$$

$$= \frac{1}{4} \Sigma \left(\frac{1}{s} - \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right) da$$

$$= \frac{1}{4} \Sigma \left\{-\frac{1}{s(s-a)} + \frac{1}{(s-b)(s-c)}\right\} a\, da$$

$$=\frac{1}{4}\Sigma\frac{s(s-a)-(s-b)(s-c)}{s(s-a)(s-b)(s-c)}a\,da$$

$$=\frac{1}{4}\Sigma\frac{s(b+c-a)-bc}{S^2}a\,da$$

$$=\frac{1}{4}\Sigma\frac{(a+b+c)(b+c-a)-2bc}{2S^2}a\,da$$

$$=\frac{1}{4}\Sigma\frac{b^2+c^2-a^2}{2S^2}a\,da$$

$$=\frac{1}{4}\Sigma\frac{2bc\cos A\,.\,a\,da}{2S^2}$$

$$=\frac{1}{4}\frac{abc}{S^2}\Sigma\cos A\,da.$$

$$\therefore\quad dS=\frac{abc}{4S}(\cos A\,da+\cos B\,db+\cos C\,dc)$$

$$=R(\cos A\,da+\cos B\,db+\cos C\,dc).$$

$\therefore$ The error $\Delta S = R\,\Sigma\cos A\,\Delta a$ approximately.

Exercises XXXIV.

1. An electric current is measured by a tangent galva-meter, the current being proportional to the tangent of the an- of deflection. If the deflection is read as 45° and an erro one per cent. is made in reading it, find approximately percentage of error in the computed value of the current. (B.A. 35)

2. The pressure p and the volume v of a gas are connec by the relation $pv^{1\cdot4}=$ constant. Find the percentage incre in the pressure corresponding to a diminution of $\frac{1}{2}$ per cent. the volume. (B.Sc. 39)

3. The angle A of a triangle ABC is found by measurem to be 63 degrees and the area is calculated by the form $\frac{1}{2}bc\sin A$. Find the percentage error in the calculated value of area, due to an error of 15 minutes in the measured value of (B.A. 37 & 61)

4. If θ be determined from v and v from ϕ by the form $\theta=\frac{\lambda}{v^2}, \phi=\frac{\mu}{v}$, ($\lambda$, μ being constants), show that the percent error in θ is nearly double the percentage error made i provided the error is small. (B.A. 46)

5. In a tangent galvanometer the tangent of the angle deflection is proportional to the current. Prove that the rel

error in the inferred value of the current due to a given error of reading is least when the deflection is 45°. (B.A. Sub. 37)

6. The formula $\frac{T_1}{T_2} = e^{M\theta}$ gives the ratio of the tensions in a belt passing round a pulley, when the belt is on the point of slipping, θ being measured in radians ; if $M = 0{\cdot}3$ and the error in $\theta = 4°$, find the percentage of error in $\frac{T_1}{T_2}$. (B.A. Sub. 35)

7. AB is a vertical pole and CD a horizontal line which when produced passes through B the foot of the pole. The tangents of the angles of elevation at C and D of the top of the pole are found to be $\frac{4}{3}$ and $\frac{3}{4}$ respectively. If CD is 35 ft., prove that an error of one minute in the determination of the elevation of D will cause an error of approximately one inch in the calculated height of the pole. (B.A. 38 M)

8. If q is calculated from the formula $q = kr^2\sqrt{h}$, show that a small percentage error in r is four times as serious as the same percentage error in h. (A.U. 59)

9. The area of a triangle is calculated from the angles A and C and the side b. If a small error δA is made in measuring A, show that the percentage error in the area is approximately $\frac{100 \sin C}{\sin A \sin (A + C)} \delta A$.

10. The base angles of a triangle are measured and found to be 67° and 82° and the base is found to be 32·4″. If the error in the length of the base does not exceed 0·1 inch, what is the upper limit to the percentage of error in the area ?

11. The work that must be done to propel a ship of displacement D for a distance S in time t is proportional to $\frac{S^2 D^{3/2}}{t^2}$. Estimate roughly the percentage increase of work necessary when the distance is increased by 1 %, the time is diminished by 1 % and the displacement of the ship is diminished by 3 %. (B.A. Sub. 50)

12. If $pv^2 = k$ and if the maximum relative error in p be not greater than 0·05 and that in v not greater than 0·025, show that the error in k may range up to 10 per cent. (B.Sc. 47 M)

13. In measuring two sides of a triangle which include an angle of 30°, one side is found to be 27 inches with a possible

error of 0·10 inch and the other 13 inches with a possible er of 0·05 inch. What is the approximate value for the larg possible error in the area of the triangle due to the errors measuring the sides ? (B.A. 52)

14. If the refractive index μ of the medium is $\dfrac{\sin x}{\sin y}$ and if error e in the measurement of x is double the error in measurement of y, find the error in calculating μ in terms x, y, e. (B.A. 45)

15. The perpendicular distance x of a point P from a fi line AB is calculatad from the length of AB $(= a)$ and the ang PAB $(= \alpha)$ and PBA $(= \beta)$. Prove that

$$x \sin(\alpha + \beta) = a \sin \alpha \sin \beta$$

and show how to find the error in x due to small errors in measurement of a, α and β.

If $\alpha = \beta = 60°$, find the percentage error in x due to error of $+1'$ in each of the measured values of α and β. (B.Sc. 38)

16. The dimensions of a cone are : radius 4 inches altitude 6 inches. What is the error in the calculation of volume of the cone if there is a shortage of 0·01 inch per in the measure used ? (B.A. Sub. 52 ; B.Sc. 63)

17. The weight of copper w deposited in t seconds a current c passing through copper sulphate is $w = cte$, where electrochemical equivalent of copper. If the current can read to a %, the weight to b %, the time to c %, find the pos percentage error in calculating e. (B.A. Sub.

18. (a) Two sides and the included angle of a triangle measured as 80", 40" and 60° with possible errors of ·001", and 1 minute respectively. Find the largest possible err the length of the third side as calculated from the observed va (B.A. Sub

(b) In measuring each of the elements b, c, $\hat{A}$ of as 5", 3" and 35°, there is known to be an error not exceeding the second decimal place. Find an approximation to the maxi possible error involved in estimating the side a in inches measurements. To what place of decimals is your approxim correct ? (B.Sc

19. If the density s of a body be inferred from its weights w, w_1 in air and water respectively, the proportional error due to small errors δw, δw_1 in these weighings is

$$\frac{\delta s}{s} = -\frac{w_1\,\delta w}{w-w_1} + \frac{\delta w_1}{w-w_1}.$$

20. The focal length of a mirror is given by the formula $\frac{1}{v} - \frac{1}{u} = \frac{2}{f}$. If equal errors δ are made in the determinations of u and v, show that the relative error $\frac{\delta f}{f}$ in the focal length is given by $\delta\left(\frac{1}{u} + \frac{1}{v}\right)$. (B.A. 39 M)

21. The resistance of a circuit was found by using the formula $C = \frac{E}{R}$. If there is a possible error of $\frac{1}{10}$ ampere in reading C and $\frac{1}{20}$ volt in reading E, what is the possible error in R if the readings are $C = 18$ amperes and $E = 110$ volts ? What is the maximum percentage error ?

22. If the deflection at the centre of an iron rod of length l and diameter d supported at its ends and loaded at the centre with a weight w is known to be proportional to $\frac{wl^3}{d^4}$, what is the percentage increase in the deflection if the load w is increased by 3 %, the length by $1\frac{1}{2}$ % and the diameter by 1 % ?

(B.A. Sub. 41)

23. The range R of a projectile which starts with a velocity v at an elevation α is given by $R = \frac{v^2 \sin 2\alpha}{g}$. Find the percentage error in R due to an error of 1 % in v and an error of $\frac{1}{2}$ % in α. (B.A. Sub. 45 ; B.Sc. 51 T.U.)

24. If $t\sqrt{\frac{g}{l}} = T\sqrt{\frac{G}{L}}$ and the values of L and G differ but slightly from those of l and g, prove that

$$\frac{T-t}{T} = \frac{1}{2}\left(\frac{L-l}{L} - \frac{G-g}{G}\right) \text{ approximately.}$$

(B.A. Sub. 47)

25. If the specific gravity of a body is determined by the formula $s = \frac{A}{A-W}$, where A is the weight of the body in air

and W is the weight in water, what is approximately th maximum error in s if A can be read within 0·01 lb. and within 0·02 lb., the actual readings being $A = 9$ lb., $W = 5$ lb. What is the maximum relative error ? (B.Sc. 32 V

26. The height of a cone is measured as 15 feet and t slant height as 25 feet ; errors of 1 per cent are to be allowed both these lengths. Find the possible error in the calculat volume. (B.Sc. 41 V

27. If a triangle ABC be slightly varied but so as to rem inscribed in the same circle, prove that

$$\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 0.$$ (B.A. 40 V

28. If the area Δ of a triangle is calculated from measu ments of one side a and the adjacent angles B and C, show th the proportional error in the area, due to small errors is given

$$\frac{\delta \Delta}{\Delta} = 2\frac{\delta a}{a} + \frac{c}{a}\frac{\delta B}{\sin B} + \frac{b}{a}\frac{\delta C}{\sin C}.$$

29. If the three sides a, b, c of a triangle are measured the angle A calculated, show that the error in A due to gi small errors δa, δb, δc in the sides is

$$\delta A = \frac{\sin A}{\sin B \sin C}\frac{\delta a}{a} - \cot C\frac{\delta b}{b} - \cot B\frac{\delta c}{c}.$$

(B.Sc. 52

30. The area Δ of a triangle is calculated from the leng of the sides a, b, c. If a be diminished and b increased by t same small amount α, prove that the consequent change in t area is given by

$$\frac{\delta \Delta}{\Delta} = \frac{2(a-b)\alpha}{c^2 - (a-b)^2}.$$ (B.Sc. 53

31. If the sides a, b, c of a triangle ABC with fixed changed by δa, δb, δc, show that

$$\delta a \cos A + \delta b \cos B + \delta c \cos C = 0.$$

32. The sides of an acute angled triangle ABC are measur Prove that the increment in A due to small increments in a, is given by the equation

$$bc \sin A\, \delta A = -a(\cos C\, \delta b + \cos B\, \delta c - \delta a).$$

Supposing that the limits of error in the length of any side $\pm \mu$ per cent, where μ is small, prove that the limits of error

are approximately

$$\pm 1{\cdot}15 \left(\frac{\mu a^2}{bc \sin A}\right) \text{ degrees.}$$

33. The height h and the semi-vertical angle α of a cone are measured and from them A, the total area of the cone, including the base is calculated. If h and α are in error by small quantities δh and $\delta\alpha$ respectively, find the corresponding error in the area. Show further that if $\alpha = \frac{\pi}{6}$, an error of $+1$ per cent. in h will be approximately compensated by an error of $-0{\cdot}33°$ in α.

34. The area S of a triangle is calculated from the lengths of the sides a, b, c. Prove that the error in S resulting from a small error in the measurement of the side c is given by

$$\Delta S \doteq \frac{S}{4}\left[\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c}\right] \Delta c,$$

where $2s = a + b + c$. (B.Sc. 53 T.U.)

35. The diameter of a circle is calculated from a segment of the circle by measuring the length a of the chord and the maximum height b of the segment. If the measurements of a and b are slightly less than the actuals to an extent of p % in each case, find the percentage error in the calculated value of the diameter. (B.A. 54 M)

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CHAPTER IX

TANGENT AND NORMAL

§ 36·1. The direction of the tangent.

We have learned in Chapter IV, Article § 23·1 that th gradient of the tangent at (x, y) to the curve $y = f(x)$ is $\frac{dy}{dx}$ ar the gradient at any point of a curve is defined as the trigon metrical tangent of the angle which the geometrical tangent the point makes with the positive direction of the x-axis. If th angle is negative, the gradient is negative; if the tangent parallel to the axis of x, the gradient is zero.

Examples.

Ex. 1. Find the angle which the tangent at (2, 4) to th curve $y = 6 + x - x^2$ makes with the x-axis.

$$\frac{dy}{dx} = 1 - 2x.$$

At (2, 4), the value of $\frac{dy}{dx} = -3$.

$$\therefore \quad \tan \Psi = -3.$$

$$\text{Hence} \quad \Psi = 160° \, 34' \text{ nearly}.$$

Ex. 2. What is the direction of the tangent at (2, 1) to th curve $x^3 + y^3 = 9$?

Differentiating the equation of the curve, we get

$$3x^2 + 3y^2 \frac{dy}{dx} = 0.$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x^2}{y^2}.$$

At the point (2, 1), the value of $\frac{dy}{dx} = -4$.

$\therefore$ The tangent at (2, 1) makes an angle $\tan^{-1}(-4)$ wi the x-axis.

If a curve passes through the origin, we can find the sha of that curve at the origin. For example,

(1) the parabola $y^2 = 4ax$ passes through the origin.

$$\text{Here } \frac{dy}{dx} = \frac{2a}{y}.$$

$\therefore$ The value of $\frac{dy}{dx}$ at (0, 0) is infinity.

$\therefore$ $\tan \Psi = \infty$, *i.e.*, $\Psi = \pi/2$.

$\therefore$ At the point (0, 0) the tangent is at right angles to the x-axis, *i.e.*, the y-axis is the tangent to the curve at the origin.

(2) For the curve $y = \frac{x^2}{1+x^2}$, $\frac{dy}{dx} = \frac{2x}{(1+x^2)^2}$.

$\therefore$ At the origin, the value of $\frac{dy}{dx} = 0$.

$$\therefore \quad \Psi = 0.$$

The curve touches the x-axis at the origin.

(3) In a curve if the value of $\frac{dy}{dx}$ at (0, 0) is 1, then $\tan \Psi = 1$, *i.e.*, $\Psi = 45°$.

The tangent to the curve at the origin bisects the angle between the coordinate axes.

Ex. 3. At which point on the curve $y = x^3 - 12x + 18$ is the tangent parallel to the x-axis ?

$$y = x^3 - 12x + 18.$$

$$\frac{dy}{dx} = 3x^2 - 12.$$

The tangent and the x-axis are parallel.

$\therefore$ $\Psi = 0$, *i.e.*, $\tan \Psi = 0$.

$\therefore$ $3x^2 - 12 = 0$

i.e., $x = \pm 2$.

The corresponding values of y are 2 and 34.

$\therefore$ The tangents at the points (2, 2), $(-2, 34)$ are parallel to the x-axis.

Ex. 4. At which point is the tangent to the curve $x^2 + y^2 = 5$ parallel to the line $2x - y + 6 = 0$?

Differentiating the equation of the curve, we get

$$2x + 2y\frac{dy}{dx} = 0.$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x}{y}.$$

At the point (x, y), the gradient of the tangent $= -\frac{x}{y}$.

The gradient of the line $2x - y + 6 = 0$ is 2.

$\therefore$ $-\frac{x}{y} = 2$, *i.e.*, $x = -2y$.

Substituting this value in $x^2+y^2=5$, we get

$$4y^2+y^2=5.$$

$$\therefore\ y=\pm 1.$$

The corresponding values of x are -2 and $+2$.

$\therefore$ At the points $(-2, 1)$, $(2, -1)$, the tangents to t curve $x^2+y^2=5$ are parallel to $2x-y+6=0$.

§ 36·2. Equations of the tangent and normal at a point of a curve.

Let the equation of the curve be $y=f(x)$ and the coor nates of P be (x_1, y_1).

Since the tangent at P makes an angle θ with the x-axis, gradient of the line is $\tan\theta$.

We have seen already that $\tan\theta=\dfrac{dy}{dx}$ at (x_1, y_1).

The value of $\dfrac{dy}{dx}$ at (x_1, y_1) is usually denoted by $\left(\dfrac{dy}{dx}\right)_1$.

$\therefore$ The equation of the tangent at P is

$$y-y_1=\tan\theta\,(x-x_1)$$

i.e., $$y-y_1=\left(\frac{dy}{dx}\right)_1(x-x_1).$$

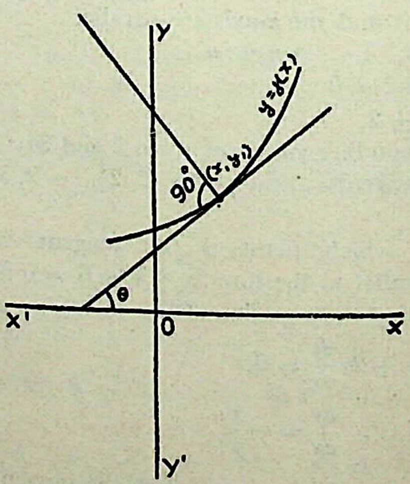

Fig. 17

The normal to a curve at any point is the straight line which passes through that point and is at right angles to the tangent the curve at that point.

Any line through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

This will be perpendicular to the tangent to the curve if

$$m\left(\frac{dy}{dx}\right)_1 = -1.$$

Hence the normal at (x_1, y_1) to the curve $y = f(x)$ is

$$(y - y_1)\left(\frac{dy}{dx}\right)_1 + (x - x_1) = 0.$$

If the equation of the curve is given in the parametric form $x = f(\theta)$ and $y = \phi(\theta)$; since

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta},$$

the equation of the tangent becomes

$$\frac{y - \phi(\theta)}{\dfrac{dy}{d\theta}} = \frac{x - f(\theta)}{\dfrac{dx}{d\theta}}$$

and the equation of the normal becomes

$$\left\{x - f(\theta)\right\}\frac{dx}{d\theta} + \left\{y - \phi(\theta)\right\}\frac{dy}{d\theta} = 0.$$

If the equation of the curve is given in the form $f(x, y) = 0$, we can calculate $\frac{dy}{dx}$ by differentiating the equation and then write down the equations of the tangent and normal at the point (x_1, y_1).

Ex. 5. Find the equation of the tangent to the curve $y = \frac{6x}{x^2 - 1}$ at the point (2, 4).

Differentiating the equation of the curve, we get

$$\frac{dy}{dx} = -6\frac{x^2 + 1}{(x^2 - 1)^2}.$$

$$\frac{dy}{dx} \text{ at } (2, 4) = -\frac{10}{3}.$$

The equation of the tangent to the curve is at (2, 4) is

$$y - 4 = -\frac{10}{3}(x - 2)$$

i.e., $10x + 3y - 32 = 0.$

Ex. 6. Find the equation of the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at } (x_1, y_1).$$

Differentiating the equation of the curve, we get

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0.$$

$$\therefore \frac{dy}{dx}=-\frac{b^2x}{a^2y}.$$

The value of $\frac{dy}{dx}$ at (x_1, y_1) is $-\frac{b^2}{a^2}\frac{x_1}{y_1}$.

$\therefore$ The equation of the tangent at the point (x_1, y_1) is

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$i.e., \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}.$$

Since the point (x_1, y_1) lies on the ellipse,

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1.$$

$\therefore$ The equation of the tangent at (x_1, y_1) on the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

Ex. 7. Find the points on the curve $y=x^4-6x^3+13x^2-10x+5$ where the tangents are parallel to $y=2x$ and prove that two of these points have the same tangent.

Let the tangent at the point (x_1, y_1) be parallel to $y=2x$.

$$\frac{dy}{dx}=4x^3-18x^2+26x-10.$$

The gradient of the tangent at (x_1, y_1) is

$$4x_1^3-18x_1^2+26x_1-10.$$

Since the tangent is parallel to $y=2x$, we get

$$4x_1^3-18x_1^2+26x_1-10=2$$

$$i.e., \quad 2x_1^3-9x_1^2+13x_1-6=0$$

$$i.e., \quad (x_1-1)(2x_1-3)(x_1-2)=0.$$

$$\therefore \quad x_1=1 \text{ or } \tfrac{3}{2} \text{ or } 2.$$

Corresponding to these values of x_1, y takes the values 3, 5, 65/16.

$\therefore$ The points on the curve where the tangents are parallel to $y=2x$ are (1, 3), (2, 5) and $\left(\frac{3}{2}, \frac{65}{16}\right)$.

The equations of the tangent at these points are respectively $y-2x-1=0$, $y-2x-1=0$ and $y-2x-\frac{17}{6}=0$.

$\therefore$ At the two points (1, 3) and (2, 5), we have the same tangent.

Ex. 8. Find the equation of the tangent at the point 'θ' on the following curve :—

$$x = a(\theta - \sin\theta);\ y = a(1 - \cos\theta).$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \cot\frac{\theta}{2}.$$

The equation of the tangent at 'θ' is

$$y - a(1 - \cos\theta) = \cot\frac{\theta}{2}\{x - a(\theta - \sin\theta)\}.$$

Simplifying, we get $y - 2a = (x - a\theta)\cot\frac{\theta}{2}$.

Note.—The equation to the tangent must in every case be simplified as much as possible.

Exercises XXXV.

1. Find the equations of the tangent and normal to each of the following curves at the points indicated :—

(1) $y = x^3$ at $\left(\frac{1}{2}, \frac{1}{8}\right)$.

(2) $x^2 + 5y^2 = 14$ where $y = 1$.

(3) $y = a\log\sin x$ at (x_1, y_1).

(4) $x = \sin t, y = \cos 2t$ at $t = \frac{\pi}{6}$.

(5) $x = a\sin^3 t, y = a\cos^3 t$ at 't'.

(6) $16x^2 + 9y^2 = 144$ at $(2, 4\sqrt{5}/3)$. (B.Sc. 60)

2. If P is the point $(at^2, 2at)$ on the parabola $y^2 = 4ax$, show that the equations of the tangent and normal at P are

$$y = \frac{x}{t} + at;\ y + tx = 2at + at^3.$$

3. If P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, find the equations of the tangent and normal to the curve at P.

4. Find the equations of the tangent and normal to the curve $y^2 = \frac{x^3}{4 - x}$ at the point $(2, -2)$. (B.A. 41 M)

5. Find the equations of the tangents drawn from the origin to the curve $y = 4x^3 - 2x^5$. (B.Sc. 42 T.U.)

6. Show that the equation to the tangent at the point (a, b) on the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ is $\frac{x}{a} + \frac{y}{b} = 2$. (B.A. Sub. 46)

7. At what point on the curve $y = \frac{20}{x}$ is the tangent par[allel] to the line $y = 15 - 2{\cdot}5x$? (B.A. Sub.

8. Find where the tangent is parallel to either coordi[nate] axis for the curve $y^3 = x^2(2a - x)$.

9. Prove that $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at point where the curve crosses the axis of y. (B.A. 36

10. Write down the equations of the tangent to the c[urve] $x^4 + a^2xy + b^3y + c^4 = 0$ at the point (h, k) on it. (B.A. 35

11. Show that the curves $y = 6 + x - x^2$, $y(x - 1) = x$ touch at the point (2, 4) and find the equation of the com[mon] tangent at that point. (B.A. Sub.

12. Show that the curves $y = x^3 + x + 1$ and $2y = x^3 +$ touch at the point (1, 3) and find the equation of the com[mon] tangent there. (B.A. Sub.

13. Show that the curves $y = x^3 - 3x^2 - 8x - 4$ $y = 3x^2 + 7x + 4$ touch at the point $(-1, 0)$. Find also equation of the common tangent. (B.A. Sub.

14. (*a*) Show that the tangents to the curve $x^3 + y^3 =$ at the points where it meets the parabola $y^2 = ax$ are paral[lel to] the axis of y. (B.Sc. 50 M ; B.A. 5

Show also that the tangents to the same curve at points where it meets $x^2 = ay$ are parallel to the axis of x.

(*b*) Prove that the tangents to $xy(y^2 - x) = a^4$ at points where it is cut by the parabola $y^2 = 2x$ are parallel to x-axis. (B.Sc.

15. Prove that the equation of the normal at the p[oint] $(a \cos^4 \theta, a \sin^4 \theta)$ on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ $16(x \cos^2 \theta - y \sin^2 \theta) = a(15 \cos 2\theta + \cos 6\theta)$. (B.A. 5

16. If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ make[s] angle ϕ with the axis of x, show that its equatio[n is] $y \cos \phi - x \sin \phi = a \cos 2\phi$. (B.Sc. 5

17. Find the equation of the tangent at any point P o[f the] curve $y^2 = x^3$. If the tangent at P intersects the curve again and the straight lines OP, OQ make angles α, β with x-axis, show that $\tan \alpha = -2 \tan \beta$. (B.Sc. 5

18. Find the equation of the tangent to the c[urve] $27ay^2 = 4(x - 2a)^3$ at the point $x = a(2 + 3t^2)$, $y =$ Show that this line is also normal to the parabola $y^2 = 4ax$ [at the] point $(at^2 - 2at)$. (B.Sc. 5

19. Trace the curve $y^2 = x(x-1)(2-x)$ and show that the tangents at the points on the curve for which $x = 1 + \frac{1}{\sqrt{3}}$ are parallel to the axis of x. (B.A. 38 M)

20. Show that the tangents at the points where the st. line $ax + hy = 0$ meets the ellipse $ax^2 + 2hxy + by^2 = 1$ are parallel to the x-axis and that the tangents at the points where the straight line $hx + by = 0$ meets the ellipse are parallel to the y-axis. (B.A. 37 M)

21. Find the coordinates of the point on the curve $x^3 + y^3 = 2ax^2$ at which the normal is parallel to the y-axis. (B.Sc. 37 M)

22. Find the equation of the tangent to the curve $y^2 = \frac{x^3}{2a-x}$ at the point (a, a) and show that this tangent meets the curve again at the point $\left(\frac{2a}{5}, -\frac{a}{5}\right)$. Prove also that the area of the triangle formed by the tangent and normal at (a, a) and the line $x = 2a$ is equal to $\frac{5a^2}{4}$. (B.A. 39 M)

23. (*a*) Find the equation of the tangent at the point (am^2, am^3) on the curve $ay^2 = x^3$.

The tangent at a point P on this curve cuts the curve again at Q. Find the relation between the values of m at P and Q. (B.A. Sub. 45)

(*b*) In the curve $27y^2 = x^3$, find the equation of the tangent at the point P (12, 8) on the curve. Verify that this tangent again cuts the curve at the point Q (3, − 1). If O is the origin, find by integration the area enclosed by the arcs OP, OQ of the curve and line PQ.

(*Hint* : Integrate with respect to y.)

24. Show that the line joining P (2, 3) on $y = \frac{2x^2 + x - 7}{x^2 - 3}$ and Q (− 1, 2) on $y + x^2 + 5x + 2 = 0$ is normal to both the curves at P and Q respectively. (B.Sc. Sub. 33)

25. If $p = x\cos\alpha + y\sin\alpha$ touches the curve

$$\left(\frac{x}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y}{b}\right)^{\frac{n}{n-1}} = 1,$$

prove that $p^n = (a\cos\alpha)^n + (b\sin\alpha)^n$.

26. The equation of a plane curve is $y^3 + x^3 - 9xy + 1$ = and (x_1, y_1) is a point on the curve at which the tanger parallel to the x-axis. Prove that at (x_1, y_1) the valu $\dfrac{d^2y}{dx^2}$ is $\dfrac{18}{27 - x_1^3}$. (B.A. 5

27. The tangent to the curve $y^2 = x(2-x)^2$ at the p (1, 1) cuts the curve at P. Find the equation of the tan at P. (B.Sc. Sub.

28. Show that the curves $y = 2\sin x - \sqrt{3}$ and $y = \dfrac{3x^2}{2\pi}$ touch each other at the point $\left(\dfrac{\pi}{3}, 0\right)$. Find the length of intercept made on the y-axis between the common tangent the common normal to the curves at this point. (B.A. Sub.

29. Find the abscissa of the point on the curve $ay^2 =$ the normal at which cuts off equal intercepts from the coord axes. (B.A. 56

§ 36·3. The following examples will illustrate the prope of the tangents and normals to the curves from their equatio

Examples.

Ex. 9. Show that the portion of the tangent i cepted between the asymptotes of the rectangular hyper $xy = c^2$ is bisected by the point of contact. Show also tha area of the triangle formed by the tangent and the asympto constant.

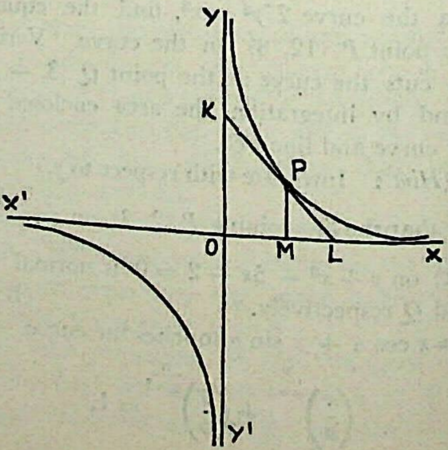

Fig. 18

The asymptotes of the rectangular hyperbola $xy = c^2$ are the coordinate axes.

Let the coordinates of the point P be (x_1, y_1).

Differentiating the equation of the curve $xy = c^2$, we get

$$x\frac{dy}{dx} + y = 0. \qquad \therefore \frac{dy}{dx} = -\frac{y}{x}.$$

$\therefore$ The equation of the tangent to the curve at P is

$$y - y_1 = -\frac{y_1}{x_1}(x - x_1), \quad i.e., \quad \frac{x}{x_1} + \frac{y}{y_1} = 2.$$

Let this tangent meet the coordinate axes in L, K.

Let the ordinate of P be MP.

To find the point where the tangent at P meets the x-axis, put $y = 0$.

$$\therefore \frac{x}{x_1} = 2, \quad i.e., \quad OL = 2x_1 = 2\,OM.$$

$$\therefore KL = 2\,KP. \quad \text{Hence } P \text{ bisects } KL.$$

To find the point where the tangent at P meets the y-axis, put $x = 0$. $\therefore \dfrac{y}{y_1} = 2$. $\therefore OK = 2y_1$.

$$\begin{aligned} \text{Area of the triangle } KOL &= \tfrac{1}{2}\,OK\,.\,OL \\ &= \tfrac{1}{2}\,.\,2y_1\,.\,2x_1 \\ &= 2x_1y_1 \\ &= 2c^2. \end{aligned}$$

$\therefore$ The area of the triangle KOL does not depend on the position of P on the curve.

Ex. 10. Show that for the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ that portion of the tangent included between the coordinate axes is constant and is equal to a.

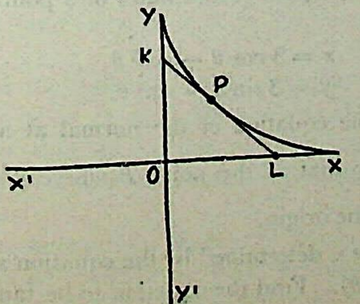

Fig. 19

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Any point on the curve can be represented as

$$(a \cos^3 \theta, a \sin^3 \theta).$$

Let the tangent at P meet the coordinate axes in L, K.

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = -\tan \theta.$$

$\therefore$ The tangent at P to the curve is

$$y - a \sin^3 \theta = -\tan \theta \,(x - a \cos^3 \theta).$$

Simplifying, $x \tan \theta + y = a \sin \theta$.

Putting, $y = 0$, the intercept OL on the x-axis is given by

$$OL = \frac{a \sin \theta}{\tan \theta} = a \cos \theta.$$

Similarly, putting $x = 0$, we get

$$OK = a \sin \theta$$
$$KL^2 = OL^2 + OK^2 = a^2$$
$$KL = a.$$

Exercises XXXVI.

1. Find the equation of the tangent to the curve $x = a \cos^4 \theta$, $y = a \sin^4 \theta$ at any point 'θ', and show that the tangent meets the axes of coordinates at two points such that the sum of their distances from the origin is constant. (B.Sc. 44 A.U.)

(*Note.*—The above curve is given by $\sqrt{x} + \sqrt{y} = \sqrt{a}$.)

2. In the catenary $y = c \cosh x/c$, prove that the perpendicular dropped from the foot of the ordinate on the tangent is constant. (B.A. 40 ...)

3. If x_1, y_1 be the parts of the axes of x and y intercepted by the tangent at any point on the curve

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1, \text{ show that } \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$$

4. The rectangular coordinates of a point on a plane curve are given by

$$x = 3 \cos \theta - \cos^3 \theta,$$
$$y = 3 \sin \theta - \sin^3 \theta.$$

Find the equation of the normal at any point on the curve and show that at the point P where $\theta = \frac{\pi}{4}$, the normal passes through the origin.

5. A curve is determined by the equation $x = a(\theta + \sin \theta)$, $y = a(1 + \cos \theta)$. Find the equation to be tangent at the p...

corresponding to a given value of θ and prove that the tangents at the points corresponding to θ, $\theta + \pi$ are at right angles to one another.

6. Find the equation of the tangent at the point 'θ' to the cycloid

$$x = a(\theta + \sin\theta),$$
$$y = a(1 - \cos\theta).$$

Show that the normal at this point goes through the point $(a\theta, 2a)$.

7. In the curve $x^m y^n = a^{m+n}$, prove that the portion of the tangent intercepted between the axes is divided at its point of contact into segments which are in a constant ratio.

(B.Sc. 43 T.U.)

8. If p and q be the lengths of the perpendiculars from the origin on the tangent and normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$, then prove that $4p^2 + q^2 = a^2$.

9. Obtain the tangent to $x^{2/3} + y^{2/3} = a^{2/3}$ at the point $(a\cos^3\theta, a\sin^3\theta)$. If the tangent meets the axes in T and T', show that the locus of the middle point of TT' is a circle.

(B.Sc. 45 T.U.)

10. Prove that all points of the curve

$$y^2 = 4a\left\{x + a\sin\frac{x}{a}\right\}$$

at which the tangent is parallel to the axis of x lie on a parabola.

(B.A. 50 M)

11. Find the equation to the locus of the intersection of perpendicular tangents to the curve $x = a\cos^3 t$, $y = a\sin^3 t$.

(B.Sc. 28 M)

12. Find for what value of n the area of the triangle included between the axes and any tangent to the curve $xy^n = a^{n+1}$ is constant.

(B.A. 38 M)

13. Prove that the equation of the tangent to the curve $x^3 + y^3 = a^3$ at the point (h, k) is $h^2x + k^2y = a^3$.

Show also that the portion of the tangent intercepted between the coordinate axes is divided at the point of contact in the ratio $h^3 : k^3$.

(B.Sc. 33 M)

14. Find the equation of the tangent at any point 't' of the curve $x = at^3$, $y = at^4$ and prove that the tangent divides the abscissa of the point of contact in the ratio $1 : 3$. (B.Sc. 52 T.U.)

C—12

15. Prove that in the tractrix

$$x = \sqrt{c^2 - y^2} + \frac{c}{2} \log \frac{c - \sqrt{c^2 - y^2}}{c + \sqrt{c^2 + y^2}}$$

the portion of the tangent at any point intercepted between t[he] point of contact and the x-axis is constant. (B.Sc. 55 [...])

§ 36·4. Angle of intersection of curves.

The angle of intersection of two curves is defined as the an[gle] between their respective tangents at the common point of inte[r]section. The angle between the two straight lines $y = m_1x +$ [...] and $y = m_2x + c_2$ is known to be $\tan^{-1}\left(\frac{m_1 \sim m_2}{1 + m_1m_2}\right)$. If we repl[ace] m_1 and m_2 in the formula by the values of $\frac{dy}{dx}$ for the two cur[ves] at their point of intersection, we get at once the angle θ at whi[ch] the curves cut there.

Examples.

Ex. 11. For the curves $x^2 = 4y$ and $y^2 = 4x$, find the an[gle] of intersection. (B.A. 41 [...])

To find the point of intersection of the curves we have [to] solve the equations $x^2 = 4y$ and $y^2 = 4x$

$$i.e., \quad \frac{x^4}{16} = 4x, \quad i.e., \quad x(x^3 - 64) = 0.$$

$$x = 0 \quad \text{or} \quad 4.$$

The corresponding values of y are 0 and $+4$.

The points of intersection are (0, 0) and (4, 4).

For the curve $x^2 = 4y$, $\frac{dy}{dx} = \frac{x}{2}$.

(i) At the point (4, 4), the value of $\frac{dy}{dx} = 2$.

For the curve $y^2 = 4x$, $\frac{dy}{dx} = \frac{2}{y}$.

At the point (4, 4), the value of $\frac{dy}{dx} = \frac{1}{2}$.

At the point (4, 4), $\tan\theta = \frac{2 - \frac{1}{2}}{1 + 2 \times \frac{1}{2}} = \frac{3}{2} \times \frac{1}{2}$.

$$\therefore \quad \theta = \tan^{-1}\left(\tfrac{3}{4}\right).$$

(ii) At (0, 0), $\frac{dy}{dx} = 0$ and ∞ and hence the values of ψ [for] the two curves are 0 and $\pi/2$.

Hence the angle of intersection is $\pi/2$, *i.e.*, the curves [cut] orthogonally.

Ex. 12. Find the angle at which the curves (1) $x^2 = ay$ and (2) $x^3 + y^3 = 3axy$ cut each other.

Substituting the values of y from $x^2 = ay$ in $x^3 + y^3 = 3axy$, we have

$$x^3 + \left(\frac{x^2}{a}\right)^3 = 3ax\frac{x^2}{a}$$

$$i.e., \quad x^3 + \frac{x^6}{a^3} = 3x^3.$$

$$x^3\left(\frac{x^3}{a^3} - 2\right) = 0.$$

$$\therefore \quad x = 0 \quad \text{or} \quad a\sqrt[3]{2}.$$

Substituting these values of x in $x^2 = ay$, we have

$$y = 0 \quad \text{or} \quad a\sqrt[3]{4}.$$

Hence the curves cut at the points $(0, 0)$ and $(a\sqrt[3]{2}, a\sqrt[3]{4})$.

Differentiating $x^2 = ay$, we get $2x = a\dfrac{dy}{dx}$.

$$\therefore \frac{dy}{dx} = \frac{2x}{a}.$$

Differentiating $x^3 + y^3 - 3axy = 0$, we get

$$\therefore \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}.$$

(*i*) The value of $\dfrac{dy}{dx}$ for the first curve at $(a\sqrt[3]{2}, a\sqrt[3]{4})$ is $2^{4/3}$ and that of $\dfrac{dy}{dx}$ for the second curve at the same point is 0.

$$\therefore \quad \tan\theta = 2^{4/3}, \quad i.e., \quad \theta = \tan^{-1}(2\sqrt[3]{2}).$$

(*ii*) At $(0, 0)$, $\dfrac{dy}{dx} = 0$ for the two curves. The two curves touch at the origin, $y = 0$ being the common tangent.

Ex. 13. Find the condition that the curves $ax^2 + by^2 = 1$, $a_1x^2 + b_1y^2 = 1$ shall cut orthogonally.

Let the curves intersect at the point whose coordinates are (x_1, y_1).

$$\therefore \quad ax_1^2 + by_1^2 - 1 = 0 \text{ and}$$

$$a_1x_1^2 + b_1y_1^2 - 1 = 0.$$

$$\therefore \quad \frac{x_1^2}{b_1 - b} = \frac{y_1^2}{a - a_1} = \frac{1}{ab_1 - a_1b}.$$

Differentiating the equations of the curve, we get

$$2ax + 2by\frac{dy}{dx} = 0, \quad i.e., \quad \frac{dy}{dx} = -\frac{ax}{by}.$$

$$2a_1x + 2b_1y\frac{dy}{dx} = 0, \quad i.e., \quad \frac{dy}{dx} = -\frac{a_1x}{b_1y}.$$

The gradients of the tangents of the two curves at the poi[nt of]

intersection are $-\frac{ax_1}{by_1}$, $-\frac{a_1x_1}{b_1y_1}$.

These curves cut each other orthogonally.

$$-\frac{ax_1}{b_1y} \times -\frac{a_1x_1}{b_1y_1} = -1.$$

$$\therefore \quad \frac{x_1^2}{y_1^2} = -\frac{bb_1}{aa_1}.$$

But the value of $\frac{x_1^2}{y_1^2}$ from (1) is $\frac{b_1 - b}{a - a_1}$.

$$\therefore \quad \frac{b_1 - b}{a - a_1} = -\frac{bb_1}{aa_1}$$

$$i.e., \quad \frac{b_1 - b}{bb_1} = \frac{a_1 - a}{aa_1}$$

$$i.e., \quad \frac{1}{b} - \frac{1}{b_1} = \frac{1}{a} - \frac{1}{a_1}$$

$$i.e., \quad \frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}.$$

Exercises XXXVII.

1. Find the angle of intersection of the circle $x^2 + y^2$ [...] and the parabola $x^2 = 2y$. (B.A. 51 [...]

2. Find the angle at which $y = 2x$ cuts the curve $x^3 + y^3 =$ [...] (B.A. Sub. 40 ; B.Sc. 51 T[...]

3. Find the angle at which the straight line $y = 2x$ inter[sects] the curve $y^2(5 - x) = x^3$.

4. Find the angles of intersection of the curves

$$y = x^2 - 3x, \; 9y = 2x^2 - 4x + 20.$$

5. Show that the curves $y = \frac{x+3}{x^2+1}$ and $y = \frac{x^2 - 7x [...]}{x - [...]}$ cut each other at the point (2, 1) at an angle of 45°. (B.Sc. Sub[...]

6. Find the angle at which the two curves $y^2 = (x$ [...] and $y = (x-1)(x-2)(x-3)$ cut at the point (2, 0). (B.Sc. Sub[...]

7. Show that the curves $x^3 - 3xy^2 = -2$, $3x^2y - y^3 =$ [...] cut orthogonally. (B.A. 5[...]

8. Show that the parabolas $y^2 = 4(x+1)$ and $y^2 = 36(9-x)$ cut orthogonally. (B.A. Sub. 47)

9. Show that the ellipse $4x^2 + 9y^2 = 72$ cuts the hyperbola $x^2 - y^2 = 5$ orthogonally at any of the points of intersection. (B.Sc. 39 M)

10. Show that the curves $x^3+y^3+x+2y=0$ and $xy+2x=y$ intersect at right angles at the origin. (B.Sc. 53 T.U.)

11. The curves $y^2 = 4ax$ and $xy = c^2$ cut at right angles. Prove that (*i*) $c^4 = 32a^4$, (*ii*) tangent and normal to either curve meet the x-axis in T and G such that $TG = 6a$.

§ 37. Subtangent and subnormal.

If the tangent and normal at the point P meet the axis of x in T and G respectively and M be the foot of the ordinate, then TM is called the *subtangent* and MG the *subnormal*.

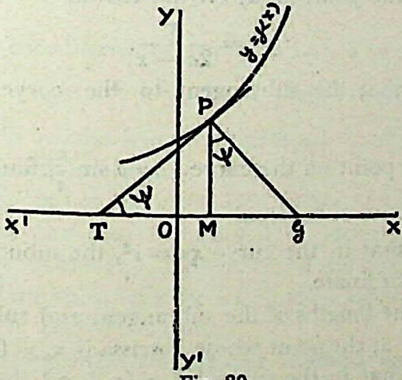

Fig. 20

$$\text{Subtangent} = TM = MP \cot \Psi = y \div \frac{dy}{dx}.$$

$$\text{Subnormal} = MG = MP \tan \Psi = y \frac{dy}{dx}.$$

$$\text{Tangent} = TP = MP \operatorname{cosec} \Psi = y\sqrt{1+\cot^2 \Psi}$$

$$= y\sqrt{1 + 1\Big/\left(\frac{dy}{dx}\right)^2}$$

$$= y\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{1/2} \div \frac{dy}{dx}.$$

$$\text{Normal} = PG = MP \sec \Psi = y\sqrt{1+\tan^2 \Psi}$$

$$= y\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{1/2}.$$

Example. Show that, in the parabola $y^2=4ax$, the subtang at any point is double the abscissa and the subnormal is consta

Differentiating the equation of the parabola $y^2 = 4ax$,

we have $2y\dfrac{dy}{dx} = 4a$ or $\dfrac{dy}{dx} = \dfrac{2a}{y}$.

$$\text{The subtangent} = \frac{y}{\frac{dy}{dx}} = \frac{y}{\frac{2a}{y}} = \frac{y^2}{2a} = 2x$$

$$= \text{double the abscissa.}$$

$$\text{The subnormal} = y\frac{dy}{dx} = y\,\frac{2a}{y} = 2a \text{ (constant).}$$

Exercises XXXVIII.

1. Find the length of the subtangent, subnormal, tan and normal at the point (a, a) on the cissoid

$$y^2 = \frac{x^3}{2a - x}.$$

2. Prove that the subtangent to the curve $y = a^x$ i constant length. (B.A. Sub.

3. At any point on the curve $y = b \sin \dfrac{x}{a}$, find the lengt the subtangent and subnormal. (B.Sc. 35

4. Prove that in the curve $xy = c^2$, the subnormal vari the cube of the ordinate. (B.A. Sub

5. Find the lengths of the subtangent and subnormal of curve $y = \sin x$ at the point whose abscissa is x. (B.A. Sub.

6. Show that in the curve $by^2 = (x + a)^3$ the square of subtangent varies as the subnormal. (B.Sc. 36

7. Prove that the subtangent for any point on the c $y = be^{x/a}$ is of constant length and the subnormal is $\dfrac{y^2}{a}$.

8. In the curve $x^m y^n = a^{m+n}$, show that the subtange any point varies as the abscissa of the point.

9. Find the subtangent and subnormal at the point d mined by 'θ' on the cycloid

$$x = a\,(\theta + \sin\theta)\,;\quad y = a\,(1 - \cos\theta).$$

10. Find the lengths of the subtangent and subnormal a point 't' on the curve

$$x = a\,(\cos t + t \sin t)$$
$$y = a\,(\sin t - t \cos t).$$ (B.A. Sub

11. In the catenary $y = c \cosh \frac{x}{c}$, prove that

(i) the length of the portion of the normal intercepted between the curve and the axis of x is $\frac{y^2}{c}$;

(ii) if along the tangent at P, a length PQ is measured equal to the ordinate at P, the locus of Q is such that at each point its subtangent is constant. (B.A. Sub. 45)

12. Show that, in the curve $y = a \log (a^2 - x^2)$, the sum of the tangent and the subtangent varies as the product of the coordinates of the point.

13. Find the maximum length of the subtangent to the curve $y(a^2 + x^2) = a^3$. (B.Sc. Sub. 54)

14. Prove that in the curve $x^{m+n} = a^{m-n} y^{2n}$, the m^{th} power of the subtangent varies as the n^{th} power of the subnormal. (T.U. 55)

15. The coordinates of any point on a curve are given by $x = \log \tan \frac{\theta}{2} + \cos \theta$, $y = \sin \theta$. Prove that the subtangent varies inversely as the length intercepted on the x-axis between the tangent and the normal. (T.U. 55)

16. Prove that in the curve $y^{n+1} = a^n x$, the subtangent varies as the abscissa and that the $(n+1)^{th}$ power of the subnormal varies inversely as the $(n-1)^{th}$ power of the abscissa. (T.U. 55)

17. Find the value of n for which the length of the subnormal of the curve $xy^n = a^{n+1}$ is constant.

§ 38. **Differential coefficient of the length of an arc of $y = f(x)$.**

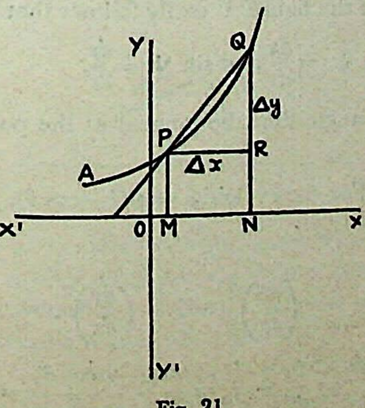

Fig. 21

Let P be any point (x, y) on the curve $y = f(x)$ and Q a point very near P, so that the coordinates of Q are $(x + \Delta x, y + \Delta y)$. Now, if s be the length of the arc AP where A is a fixed point on the curve, then $s + \Delta s$ is the length of the arc AQ, so that arc $PQ = \Delta s$.

As Δx tends to zero, Q approaches P and the chord PQ and arc PQ become almost equal. Thus the ultimate ratio of the arc PQ to the chord PQ as $\Delta x \to 0$ is unity.

Now from the right-angled triangle PQR,

$$(\text{chord } PQ)^2 = PR^2 + QR^2$$

i.e., $$(\text{chord } PQ)^2 = (\Delta x)^2 + (\Delta y)^2$$

i.e., $$\left(\frac{\text{chord } PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

i.e., $$\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \cdot \left(\frac{\text{arc } PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

Taking the limits as $\Delta x \to 0$, we get

$$\underset{\Delta x \to 0}{\text{Lt}} \frac{\text{chord } PQ}{\text{arc } PQ} = 1, \quad \underset{\Delta x \to 0}{\text{Lt}} \frac{\text{arc } PQ}{\Delta x} = \frac{ds}{dx} \text{ and } \underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

$$\therefore \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$$

In the same way it may be shown that

$$\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2.$$

Note.—From the figure, it easily follows that

$$\cos \psi = \frac{dx}{ds} \text{ and } \sin \psi = \frac{dy}{ds},$$

where ψ is the angle that the tangent at the point (x, y) makes with OX.

Example. For the cycloid $x = a(1 - \cos \theta)$, $y = a(\theta + \sin \theta)$, find $\frac{ds}{dx}$.

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a(1 + \cos \theta)}{a \sin \theta} = \cot \frac{\theta}{2}.$$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \cot^2\frac{\theta}{2} = \operatorname{cosec}^2\frac{\theta}{2}.$$

$$\frac{ds}{dx} = \operatorname{cosec}\frac{\theta}{2}.$$

Exercises XXXIX.

1. For the parabola $y^2 = 4ax$, prove that

$$\frac{ds}{dx} = \left(1 + \frac{a}{x}\right)^{1/2}.$$

2. Find $\frac{ds}{dx}$ in each of the following curves :—

(1) $x^{2/3} + y^{2/3} = a^{2/3}$.

(2) $x^{1/2} + y^{1/2} = a^{1/2}$.

(3) $y = a \cosh \frac{x}{a}$.

3. For the ellipse $x = a \cos \theta$, $y = b \sin \theta$, prove that $\frac{ds}{d\theta} = a\,(1 - e^2 \cos^2 \theta)^{1/2}$ where $b^2 = a^2\,(1 - e^2)$.

4. Show that in the cycloid $x = a\,(\theta + \sin \theta)$, $y = a\,(1 - \cos \theta)$, $\frac{ds}{d\theta} = 2a \cos\frac{\theta}{2}$ and $\frac{ds}{dy} = \sqrt{\frac{2a}{y}}$.

§ 39·1. Polar coordinates.

The position of a point P on a plane can be indicated by stating (1) its distance r from a fixed point O and (2) the inclination θ of OP to a fixed straight line through O ; r and θ are called the polar coordinates of P, r is called the radius vector and θ the vectorial angle. O the pole and OA the initial line.

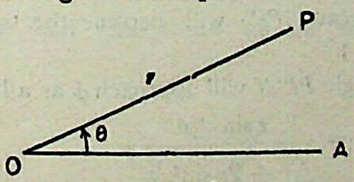

Fig. 22

r is considered to be positive when measured away from O along the line bounding the vectorial angle and θ is considered to be positive when measured in the anti-clockwise direction. It is usual to regard θ as the independent variable.

When converting polar coordinates to cartesian or *vice versa*, it is customary to take the pole as the origin and the initial line

as the x-axis. Then the formulae for conversion are $x = r\cos\theta$, $y = r\sin\theta$.

§ 39·2. Angle between the radius vector and the tangent.

Let P, P' be two neighbouring points on a curve and (r, θ) be the polar coordinates of P and $r + \Delta r$, $\theta + \Delta\theta$ those

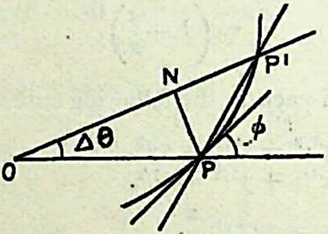

Fig. 23

of P'. If we join PP' and draw PN perpendicular to OP', we have

$$PN = OP \sin \angle PON$$
$$= r \sin \Delta\theta.$$
$$P'N = OP' - ON = r + \Delta r - r\cos\Delta\theta$$
$$= \Delta r + r(1 - \cos\Delta\theta)$$
$$= \Delta r + 2r\sin^2\frac{\Delta\theta}{2}.$$

Denote by ϕ the angle between the radius vector OP and the tangent at P. If we now let $\Delta\theta$ approach the limit zero, then

(1) the point P' will approach P;

(2) the secant PP' will become the tangent PT in the limiting positions;

(3) the angle $PP'N$ will approach ϕ as a limit.

$$\tan PP'O = \frac{PN}{P'N} = \frac{r\sin\Delta\theta}{\Delta r + 2r\sin^2\frac{\Delta\theta}{2}}$$

$$= r\frac{\dfrac{\sin\Delta\theta}{\Delta\theta}}{\dfrac{\Delta r}{\Delta\theta} + \dfrac{r\,.\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\,.\sin\dfrac{\Delta\theta}{2}}.$$

$$\operatorname*{Lt}_{\Delta\theta\to 0}\frac{\sin\Delta\theta}{\Delta\theta}=1, \qquad \operatorname*{Lt}_{\Delta\theta\to 0}\sin\frac{\Delta\theta}{2}=0,$$

$$\operatorname*{Lt}_{\Delta\theta\to 0}\frac{\Delta r}{\Delta\theta}=\frac{dr}{d\theta}.$$

$$\therefore\quad \tan\phi=\operatorname*{Lt}_{\Delta\theta\to 0}\tan PP'O$$

$$=r\cdot\frac{1}{\frac{dr}{d\theta}+r\cdot 1\cdot 0}=r\cdot\frac{d\theta}{dr}.$$

Example. Find the angle at which the radius vector cuts the curve $\frac{l}{r}=1+e\cos\theta$.

Let ϕ be the angle between the radius vector and the tangent at the point at which the radius vector meets the curve.

$$\therefore\quad \tan\phi=r\frac{d\theta}{dr}.$$

Differentiating $\frac{l}{r}=1+e\cos\theta$ with respect to θ, we get

$$-\frac{l}{r^2}\frac{dr}{d\theta}=-e\sin\theta.$$

$$\therefore\quad \frac{dr}{d\theta}=\frac{er^2}{l}\sin\theta.$$

$$\therefore\quad \tan\phi=\frac{rl}{e\sin\theta\cdot r^2}=\frac{l}{r\cdot e\sin\theta}$$

$$=\frac{1+e\cos\theta}{e\sin\theta}.$$

$$\therefore\ \phi=\tan^{-1}\left(\frac{1+e\cos\theta}{e\sin\theta}\right).$$

§ 39·3. To find the slope of the tangent in polar coordinates.

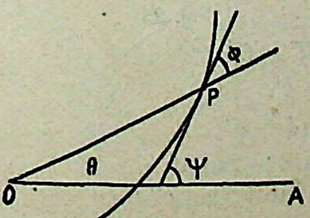

Fig. 24

Let OA be the initial line, O the pole, the polar coordinates of P, (r, θ) and the tangent at P make an angle ψ with OA.

$$\Psi = \phi + \theta.$$

$$\tan \Psi = \tan (\phi + \theta) = \frac{\tan \phi + \tan \theta}{1 - \tan \phi \tan \theta}.$$

If the equation of the curve is known, $\tan \phi$ can be calcul[ated] and hence we can easily find $\tan \Psi$.

Example. Find the slope of the tangent with the ini[tial] line for the cardioid $r = a(1 - \cos \theta)$ at $\theta = \frac{\pi}{6}$.

$$\cot \phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{a(1-\cos\theta)} . a \sin\theta = \cot\frac{\theta}{2}.$$

$$\therefore \quad \phi = \frac{\theta}{2}.$$

$$\therefore \quad \Psi = \theta + \phi = 3\theta/2.$$

If $\theta = \frac{\pi}{6}$, $\Psi = 3\pi/12 = \pi/4$.

Slope of the tangent $= \tan \Psi = 1$.

§ 39·4. To find the angle of intersection of two cur[ves] C and C′ whose equations are given in polar coordinat[es].

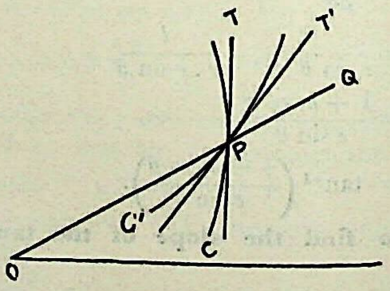

Fig. 25

$$T'PT = QPT - QPT'$$

i.e., $\Psi = \phi - \phi'$.

$$\tan \Psi = \tan(\phi - \phi')$$

$$= \frac{\tan\phi - \tan\phi'}{1 + \tan\phi . \tan\phi'}.$$

Since the equations of the curves are known, the coordi[nates] of the point of intersection P can be found and then $\tan \phi$ [and] $\tan \phi'$ can be easily calculated from the equations of the cu[rves].

Example. Find the angle of intersection of the cardioids $r = a(1 + \cos\theta)$ and $r = b(1 - \cos\theta)$.

Let PT and PT' be the tangents to the curves $r = a(1 + \cos\theta)$ and $r = b(1 - \cos\theta)$ respectively at their point of intersection P. Let O be the pole and let OP be the radius vector to the point P.

Let ϕ and ϕ' be the angles which the radius vector to the point of intersection of the curves, makes with the tangents to the curves at that point.

It is required to find $\phi \sim \phi'$.

$$r = a(1 + \cos\theta).$$

$$\frac{dr}{d\theta} = -a\sin\theta.$$

$$\tan\phi = r \cdot \frac{d\theta}{dr} = \frac{a(1 + \cos\theta)}{(-a\sin\theta)} = -\cot\theta/2.$$

$$\therefore \quad \phi = \frac{\pi}{2} + \frac{\theta}{2}.$$

Also in the case of the curve $r = b(1 - \cos\theta)$,

$$\frac{dr}{d\theta} = b\sin\theta.$$

$$\tan\phi' = \frac{b(1 - \cos\theta)}{b\sin\theta} = \tan\theta/2.$$

$$\therefore \quad \phi' = \theta/2.$$

$$\therefore \quad \phi - \phi' = \frac{\pi}{2} + \frac{\theta}{2} - \frac{\theta}{2} = \frac{\pi}{2}.$$

Hence the curves cut each other at right angles at P.

§ 39·5. Polar subtangent and polar subnormal.

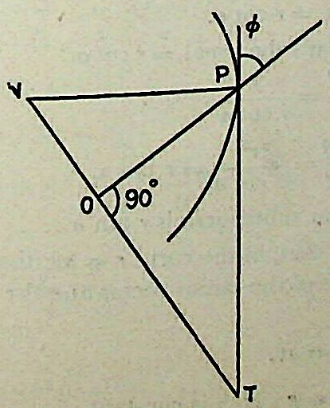

Fig. 26

Draw a line NT through the pole perpendicular to the rad vector of the point P on the curve. If PT is the tangent and P the normal to the curve at P, then

OT = length of polar subtangent.

ON = length of polar subnormal of the curve at P.

Polar subtangent $= OT = OP \tan\phi$

$$= r \cdot r\frac{d\theta}{dr} = r^2\frac{d\theta}{dr}.$$

Polar subnormal $= ON$

$$= OP \tan \angle OPN$$
$$= OP \tan (\angle TPN - \angle TPO)$$
$$= OP \tan\left(\frac{\pi}{2} - \phi\right)$$
$$= r\cot\phi = \frac{r}{\tan\phi} = \frac{r}{r\dfrac{d\theta}{dr}}$$
$$= \frac{dr}{d\theta}.$$

Examples.

Ex. 1. Show that in the curve $r = e^{\theta \cot a}$

(1) the polar subtangent $= r \tan a$

(2) the polar subnormal $= r \cot a$.

Here $r = a\, e^{\theta \cot a}$.

$$\therefore \frac{dr}{d\theta} = a\, e^{\theta \cot a} . \cot a$$
$$= r \cot a.$$

Hence the polar subnormal $= r \cot a$.

$$\text{Also } \frac{d\theta}{dr} = \frac{1}{r \cot a}.$$

$$\therefore r^2\frac{d\theta}{dr} = \frac{r^2}{r\cot a} = r \tan a.$$

Hence the polar subtangent is $r \tan a$.

Ex. 2. Show that, in the curve $r = a\theta$, the polar subtangent varies as the square of the radius vector and the polar subnormal is constant.

Now $r = a\theta$.

$$\therefore \frac{dr}{d\theta} = a \text{ which is constant.}$$

Hence the polar subnormal is constant.

Again $\dfrac{d\theta}{dr}=\dfrac{1}{a}$.

$\therefore\quad r^2\dfrac{d\theta}{dr}=\dfrac{r^2}{a}$.

Thus the polar subtangent varies as r^2.

§ 39·6. The length of arc in polar coordinates.

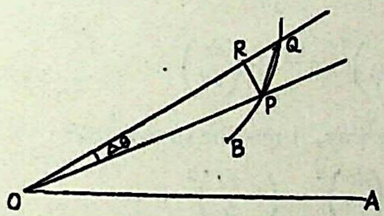

Fig. 27

Let the coordinates of a point P on the curve be r and θ so that $OP=r$ and $\angle AOP=\theta$.

Let the coordinates of a point Q on the curve very close to P be $r+\Delta r$ and $\theta+\Delta\theta$, so that $OQ=r+\Delta r$, $\angle QOA=\theta+\Delta\theta$ and $\angle POR=\Delta\theta$.

Let s be the length of the arc BP where B is a fixed point on the 'curve ; then the length of the arc BQ is $s+\Delta s$ and the arc PQ is Δs.

Now $\quad PR=OP\sin\Delta\theta=r\sin\Delta\theta \qquad (1)$

and $OR=OP\cos\Delta\theta=r\cos\Delta\theta$.

$$\begin{aligned}\text{Also } QR&=r+\Delta r-r\cos\Delta\theta\\&=r(1-\cos\Delta\theta)+\Delta r\\&=2r\sin^2\frac{\Delta\theta}{2}+\Delta r.\end{aligned}$$

$$\begin{aligned}PQ^2&=PR^2+RQ^2\\&=(r\sin\Delta\theta)^2+\left\{2r\sin^2\frac{\Delta\theta}{2}+\Delta r\right\}^2\end{aligned}$$

$$\therefore\quad\left(\frac{\text{chord }PQ}{\Delta\theta}\right)^2=\left(r\frac{\sin\Delta\theta}{\Delta\theta}\right)^2+\left\{\frac{2r\sin^2\frac{\Delta\theta}{2}}{\Delta\theta}+\frac{\Delta r}{\Delta\theta}\right\}^2$$

$$=r^2\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2+\left\{\frac{r\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\sin\frac{\Delta\theta}{2}+\frac{\Delta r}{\Delta\theta}\right\}^2$$

Passing to the limit as $\Delta\theta$ tends to zero, we get

$$\frac{\sin\Delta\theta}{\Delta\theta}\to 1,\ \frac{\Delta r}{\Delta\theta}\to\frac{dr}{d\theta},\ \sin\frac{\Delta\theta}{2}\to 0.$$

$$\therefore\quad \frac{\text{chord } PQ}{\Delta\theta}=\frac{\text{chord } PQ}{\text{arc } PQ}\cdot\frac{\text{arc } PQ}{\Delta\theta}$$

$$\to 1\cdot\frac{ds}{d\theta}.$$

$$\therefore\ \left(\frac{ds}{d\theta}\right)^2=r^2+\left(\frac{dr}{d\theta}\right)^2.$$

In the same way, it may be shown that

$$\left(\frac{ds}{d\theta}\right)^2=\left(r\frac{d\theta}{dr}\right)^2+1.$$

It is easily seen that $\cos\phi=\dfrac{dr}{ds}$ and $\sin\phi=r\dfrac{d\theta}{ds}$.

Example. Find $\dfrac{ds}{d\theta}$ and $\dfrac{ds}{dr}$ for the cardioid

$$r=a\,(1-\cos\theta).$$

Differentiating the above equations,

$$\therefore\ \frac{dr}{d\theta}=-a\sin\theta.$$

$$\text{Also}\quad r\frac{d\theta}{dr}=\frac{a\,(1+\cos\theta)}{-a\sin\theta}=-\cot\frac{\theta}{2}.$$

$$\left(\frac{ds}{dr}\right)^2=1+\left(r\frac{d\theta}{dr}\right)^2=1+\cot^2\frac{\theta}{2}=\operatorname{cosec}^2\frac{\theta}{2}.$$

$$\therefore\ \frac{ds}{dr}=\operatorname{cosec}\frac{\theta}{2}.$$

$$\left(\frac{ds}{d\theta}\right)^2=r^2+\left(\frac{dr}{d\theta}\right)^2$$

$$=a^2\,(1+\cos\theta)^2+a^2\sin^2\theta$$

$$=4a^2\cos^2\theta/2.$$

$$\therefore\ \frac{ds}{d\theta}=2a\cos\frac{\theta}{2}.$$

Exercises XL.

1. Find the slopes of the following curves at the points specified :—

(1) $r = a(1 - \cos\theta)$ at $\theta = \frac{\pi}{2}$.

(2) $r = a \sin 2\theta$,, $\theta = \frac{\pi}{4}$.

(3) $r = e^{\theta}$,, $\theta = 0$.

(4) $r = a \sin 3\theta$,, pole.

(5) $r = a^{\theta}$,, $\theta = \frac{\pi}{2}$.

2. Find ϕ in terms of θ for the curve $r^2 = a^2 \cos 2\theta$.

3. Show that the parabolas

$$r = a \sec^2 \frac{\theta}{2} \text{ and } r = b \operatorname{cosec}^2 \frac{\theta}{2}$$

intersect at right angles.

4. Prove that the spiral $r = a\theta$ and the reciprocal spiral $r = a/\theta$ intersect at right angles.

5. Show that the rectangular hyperbola $r^2 \sin 2\theta = a^2$, $r^2 \cos 2\theta = b^2$ intersect at right angles.

6. Find the angle of intersection of the curves

(i) $r = a \sin 2\theta$ and $r = a \cos 2\theta$.

(ii) $r = \dfrac{a}{1 + \cos\theta}$ and $r = \dfrac{b}{1 - \cos\theta}$. (B.Sc. 66 M)

7. Prove that the point excluding the pole itself, on the cardioid $r = a(1 + \cos\theta)$, where the tangent is parallel to the initial line is at a distance $3a/2$ from the pole. (B.Sc. 47 M)

8. Find the lengths of subtangent, subnormal, tangent and normal in the logarithmic spiral $r = a\theta$.

9. In the curve $r \cos 2\theta = a$, the polar subtangent $= \frac{1}{2} a \operatorname{cosec} 2\theta$ and the polar subnormal $= 2a \sec 2\theta \tan 2\theta$.

10. Show that in the curve $r\theta = a$, the polar subtangent is constant.

11. Show that the tangent at the point where $\theta = \frac{\pi}{6}$ on the curve $r = a \cos 2\theta$ meets the initial line at a distance of $a/\sqrt{3}$ from the pole.

12. Find the angle ϕ in the case of the curve $r^n = a^n \sec(n\theta + \alpha)$ and prove that the curve is intersected by the curve $r^n = b^n \sec(n\theta + \beta)$ at an angle which is independent of a and b.

13. In the cardioid $r = a\,(1-\cos\theta)$, show that $\dfrac{ds}{d\theta} = 2a\sin$ and p varies as $(\sqrt{r})^3$ where p is the length of the perpen-cular from the pole on the tangent.

14. In the curve $r = a\theta$, show that $\dfrac{ds}{dr} = \dfrac{\sqrt{r^2+a^2}}{a}$.

15. In the curve $r = \dfrac{a}{\theta}$, $\dfrac{ds}{d\theta} = \dfrac{\sqrt{r^2+a^2}}{r}$.

16. Calculate $\dfrac{ds}{d\theta}$ for the following curves :—

(*i*) $r = \log\sin 3\theta$.
(*ii*) $r = \frac{1}{2}\sec^2\theta$.

17. In the curve $r^2 = a^2\cos 2\theta$, prove that $\dfrac{ds}{d\theta} = \dfrac{a^2}{r}$.

18. For the curve $r^n = a^n\cos n\theta$, prove that

$$a^{2n}\frac{d^2r}{ds^2} + n\,.\,r^{2n-1} = 0.$$

19. Prove that the normal at any point (r, θ) to the cu $r^n = a^n\cos n\theta$ makes an angle $(n+1)\,\theta$ with the initial line.

20. The tangent at any point of the cardioid $r = a\,(1+\cos$ whose vectorial angle is 2α, meets the curve again at a p whose vectorial angle is 2β ; show that

$$\cos\,(2\beta-\alpha) + 2\cos\alpha = 0.$$

21. If the tangents at P, Q to $r = a\,(1-\cos\theta)$ are para and if O is the pole, prove that $\angle POQ = \frac{2}{3}\pi$.

CHAPTER X

CURVATURE OF PLANE CURVES

§ 40·1. A curve has a definite direction at every point on it. At any particular point, the direction of the curve is the same as that of the tangent to the curve at the point. The direction usually changes from point to point and the tangent line rotates as the point moves along the curve.

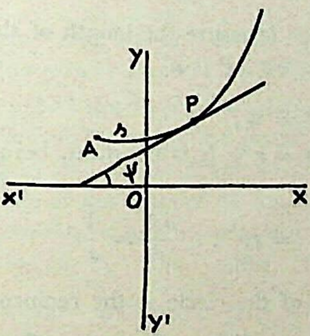

Fig. 28

Let s denote the length of arc AP measured from some fixed point A on the curve and Ψ, the angle which the tangent makes with the x-axis. As P moves along the curve, s and Ψ vary and the rate at which Ψ increases relative to s, *i.e.*, $\frac{d\Psi}{ds}$ is called the *curvature* of the curve at the point P. Its value does not depend on the position of the point A or of the line Ox from which s and Ψ are measured but its sign depends on the sense in which s is measured. From the definition of curvature at P, it is easily seen that curvature is the rate of change of the direction of the tangent at P. Roughly, we can say that the curvature is the rate at which the curve 'curves' and its sign indicates the direction in which the tangent is turning as s increases.

Let ABC be any given circle of radius r. Draw AQ the tangent at A. Let O be the centre of the circle ; join OA. Select

any point P on the circle. Draw PM the tangent at P cu[t]

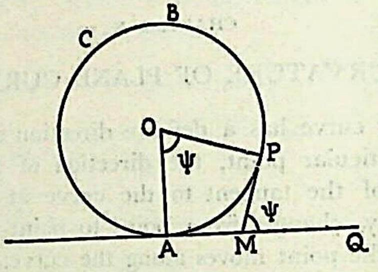

Fig. 29

AQ at an angle Ψ. Measure the length of the arc of the ci[rcle] from A, so that the arc AP is s.

As $\angle AOP = \Psi$,

$s = r\,\Psi$ (r is constant, being the radius of ci[rcle])

$$\frac{ds}{d\Psi} = r. \quad \therefore \quad \frac{d\Psi}{ds} = \frac{1}{r}$$

i.e., curvature of the circle is the reciprocal of its radi[us].

§ 40·2. Circle, radius and centre of curvature.

Let P and Q be two points on a plane curve, Ψ, $\Psi + \Delta\Psi$ the angles which the tangents at P and Q [make] with the x-axis ; s the arc measured from some fixed point [on] the curve up to P and Δs the [arc] PQ. Let the normals at P a[nd Q] intersect at C'.

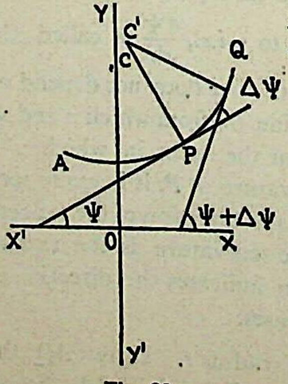

Fig. 30

From the figure, it easily f[ollows] that $\angle PC'Q = \Delta\Psi$.

$$\frac{PC'}{\sin PQC'} = \frac{\text{chord } PQ}{\sin\ PC'Q}$$

$$= \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{\text{arc } PQ}{\sin\ PC'Q}$$

$$= \left(\frac{\text{chord } PQ}{\text{arc } PQ}\right) \cdot \frac{\Delta s}{\sin \Delta\Psi}$$

$$= \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{\Delta s}{\Delta\Psi} \cdot \frac{\Delta\Psi}{\sin \Delta\Psi}$$

Now the limit of $\angle PQC'$ as Q tends to P is $90°$ and also

$$\text{Lt} \frac{\text{chord } PQ}{\text{arc } PQ} = 1.$$

$$\underset{\Delta\Psi\to 0}{\text{Lt}} \frac{\Delta s}{\Delta\Psi} = \frac{ds}{d\Psi} \text{ and Lt } \frac{\sin \Delta\Psi}{\Delta\Psi} = 1.$$

$\therefore$ As Q tends to P, the limit of PC' is $\dfrac{ds}{d\Psi}$.

Let the limiting position of C' be C.

Then $PC = \dfrac{ds}{d\Psi}$, *i.e.*, $\dfrac{1}{PC} = \dfrac{d\Psi}{ds}$.

$PC = \dfrac{ds}{d\Psi}$ is called the *radius of curvature at P*.

The circle whose centre is C and radius PC has therefore the same tangent and the same curvature as the curve has at P.

This circle is called the *circle of curvature at P*. So it can be defined as that circle which touches the given curve at the point, has a radius equal to the radius of curvature at the point and lies on the same side of the tangent as the curve. Its radius is PC, the radius of curvature and its centre is C, the centre of curvature at the point P. The radius of curvature is often denoted by ρ and so the curvature is $\dfrac{1}{\rho}$.

§ 40·3. Cartesian formula for the radius of curvature.

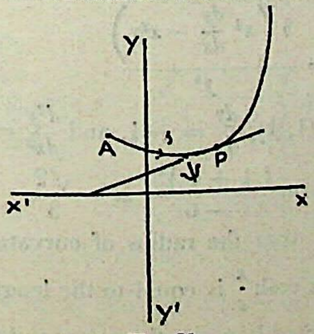

Fig. 31

We know that $\dfrac{dy}{dx} = \tan \Psi$.

$$\therefore \frac{d^2y}{dx^2} = \sec^2 \Psi \cdot \frac{d\Psi}{dx} = \sec^2 \Psi \frac{d\Psi}{ds} \frac{ds}{dx}.$$

$$\therefore \quad \frac{ds}{d\Psi} = \frac{\sec^3 \Psi}{\frac{d^2y}{dx^2}} \text{ as } \frac{dx}{ds} = \cos \Psi \text{ by § 38}$$

$$= \frac{(1+\tan^2 \psi)^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}.$$

$$\therefore \quad \rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}.$$

Examples.

Ex. 1. What is the radius of curvature of the curve $x^4+y^4=2$ at the point (1, 1) ?

Differentiating the above equation, we get

$$4x^3+4y^3\frac{dy}{dx}=0.$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x^3}{y^3}.$$

Differentiating this once again, we get

$$\frac{d^2y}{dx^2} = \frac{3\left(x^3\frac{dy}{dx}-x^2y\right)}{y^4}.$$

At the point (1, 1), $\frac{dy}{dx} = -1$, and $\frac{d^2y}{dx^2} = -6$.

$$\therefore \quad \rho = \frac{\{1+1\}^{3/2}}{-6} = -\frac{\sqrt{2}}{3}.$$

Ex. 2. Show that the radius of curvature at any point of the catenary $y = c\cosh\frac{x}{c}$ is equal to the length of the portion of the normal intercepted between the curve and the axis of x.

(B.Sc. 51 ; B.A. 51 M)

Hence $\frac{dy}{dx} = \sinh\frac{x}{c}$ and hence

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2} = \left(1+\sinh^2\frac{x}{c}\right)^{3/2} = \cosh^3\frac{x}{c}.$$

Also $\dfrac{d^2y}{dx^2}=\dfrac{1}{c}\cosh\dfrac{x}{c}$.

Here $\rho=\dfrac{\cosh^3\dfrac{x}{c}}{\dfrac{1}{c}\cosh\dfrac{x}{c}}=c\cosh^2\dfrac{x}{c}=\dfrac{y^2}{c}$.

Again at any point (x, y) the normal

$$=y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{1/2}=y\cosh\frac{x}{c}=\frac{y^2}{c}.$$

$\therefore$ Radius of curvature = length of the normal.

Ex. 3. If a curve is defined by the parametric equations $x=f(\theta)$ and $y=\phi(\theta)$, prove that the curvature

$$\frac{1}{\rho}=\frac{x'y''-y'x''}{(x'^2+y'^2)^{3/2}}$$

where dashes denote differentiation with respect to θ.

$$\frac{dy}{dx}=\frac{dy}{d\theta}\div\frac{dx}{d\theta}=\frac{y'}{x'}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{y'}{x'}\right)=\frac{d}{d\theta}\left(\frac{y'}{x'}\right)\cdot\frac{d\theta}{dx}$$

$$=\frac{y''x'-y'x''}{x'^2}\cdot\frac{1}{x'}$$

$$=\frac{y''x'-y'x''}{x'^3}.$$

$$\therefore\ \frac{1}{\rho}=\frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}=\frac{y''x'-y'x''}{x'^3\left\{1+\dfrac{y'^2}{x'^2}\right\}^{3/2}}$$

$$=\frac{x'y''-y'x''}{\{x'^2+y'^2\}^{3/2}}.$$

Ex. 4. Prove that the radius of curvature at any point of the cycloid $x=a(\theta+\sin\theta)$ and $y=a(1-\cos\theta)$ is $4a\cos\dfrac{\theta}{2}$.

From the given equations,

$$\frac{dx}{d\theta}=a(1+\cos\theta)\ ;\quad \frac{dy}{d\theta}=a\sin\theta.$$

$$\frac{d^2x}{d\theta^2}=-a\sin\theta\ ;\quad \frac{d^2y}{d\theta^2}=a\cos\theta.$$

Substituting these values in the formula obtained in the previous example, we get

$$\frac{1}{\rho}=\frac{a\,(1+\cos\theta)\,a\cos\theta-a\sin\theta\,(-a\sin\theta)}{\{a^2(1+\cos\theta)^2+a^2\sin^2\theta\}^{3/2}}$$

$$=\frac{a^2\,(1+\cos\theta)}{a^3\{\,2\,(1+\cos\theta)\,\}^{3/2}}$$

$$=\frac{2\cos^2\theta/2}{a\,(4\cos^2\theta/2)^{3/2}}=\frac{1}{4a\cos\theta/2}.$$

$\therefore\ \rho=4a\cos\theta/2.$

Ex. 5. Find ρ at the point 't' of the curve

$$x=a(\cos t+t\sin t)\ ;\ y=a(\sin t-t\cos t).$$

$$\frac{dx}{dt}=a\,(-\sin t+\sin t+t\cos t)=at\cos t.$$

$$\frac{dy}{dt}=a\,(\cos t-\cos t+t\sin t)=at\sin t.$$

$$\therefore\ \frac{dy}{dx}=\tan t.$$

Differentiating with respect to x,

$$\frac{d^2y}{dx^2}=\frac{d}{dt}(\tan t)\,\frac{dt}{dx}=\sec^2 t\,\frac{1}{at\cos t}=\frac{at}{\cos^3 t}.$$

$$\therefore\ \rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}=\frac{\left(1+\tan^2 t\right)^{3/2}}{\frac{at}{\cos^3 t}}=at.$$

(The formula of Ex. 3 can also be employed.)

Exercises XLI.

1. Find the radius of curvature for the curves

(a) $y=e^x$ at the point where it crosses the y-axis. (B.Sc. 50 A()

(b) $\sqrt{x}+\sqrt{y}=1$ at $(\frac{1}{4},\frac{1}{4})$. (B.Sc. 50 A()

(c) $y^2=x^3+8$ at the point $(-2, 0)$. (B.Sc. 32 A()

(d) $xy=30$ at the point $(3, 10)$. (B.Sc. 35 A()

(e) $(x^2+y^2)^2=a^2(y^2-x^2)$ at the point $(0, a)$. (B.A. 50 A()

(f) $xy^3=a^4$ at (a, a).

(g) $x^3+y^3=3axy$ at the point $x=y=3a/2$.

(h) $y=4\sin x-\sin 2x$ at the point where $x=\pi/2$.

(*i*) $y = \dfrac{\log x}{x}$ at $x = 1$. (B.E. 45 A.U.)

(*j*) $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 2$ at $(3, 4)$. (B.Sc. Comp. Math. 59)

2. In the ellipse given by $x = a \cos \theta, y = b \sin \theta$

(*i*) find the radius of curvature at $\left(\dfrac{a}{\sqrt{2}}, \dfrac{b}{\sqrt{2}}\right)$;

(*ii*) show that ρ, at any point P, is $\dfrac{CD^3}{ab}$ where CD is the semi-conjugate diameter to CP;

(*iii*) show that the radius of curvature at P of the ellipse multiplied by the cube of the perpendicular from the centre on the tangent at P is constant; (B.A. 51 M)

(*iv*) show that $(\rho^{2/3} + \rho'^{2/3})(ab)^{2/3} = a^2 + b^2$, where ρ and ρ' are the radii of curvature at the extremities of two conjugate diameters of an ellipse.

3. Find the radius of curvature at any point P of the parabola given by $x = at^2, y = 2at$.

Show that it is $2\dfrac{SP^{3/2}}{\sqrt{a}}$, where S is the focus of the parabola; it is also $2a \operatorname{cosec}^3 \psi$.

4. Prove that the radius of curvature at a point $(a \cos^3 \theta, a \sin^3 \theta)$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is

$$3a \sin \theta \cos \theta.$$

5. If the coordinates of a point on a curve be given by the equations

$$x = c \sin 2\theta\,(1 + \cos 2\theta)$$
$$y = c \cos 2\theta\,(1 - \cos 2\theta),$$

show that the radius of curvature at the point 'θ' is $4c \cos 3\theta$.

6. Prove that the radius of curvature at the point 'θ' on the curve

$$x = 3a \cos \theta - a \cos 3\theta,$$
$$y = 3a \sin \theta - a \sin 3\theta \text{ is } 3a \sin \theta.$$

7. Prove that the radius of curvature at any point of the parabola $x^2 = 4ay$ is $2a \sec^3 \theta$, where θ is the angle between the tangent at the point and the tangent at the vertex. (B.A. Sub. 39)

8. Show that the radius of curvature of the curve $y^2 = \dfrac{a^2(a-x)}{x}$ at $(a, 0)$ is $\frac{1}{2}a$. (B.Sc. 38 M)

Hint: Use the formula $\rho = \left\{1 + \left(\dfrac{dx}{dy}\right)^2\right\}^{3/2} \Big/ \dfrac{d^2x}{dy^2}$.

9. Find the expression for the radius of curvature at any point of the curve $x^m + y^m = 1$ and interpret the result for the cases $m = 1$ and $m = 2$. (B.A. 39 M)

10. Show that the curves $y = \dfrac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$ and $y = \dfrac{a}{2}\left(2 + \dfrac{x^2}{a^2}\right)$ have the same curvature at $(0, a)$ and find the radius of curvature at this point. (B.A. Sub. 46)

11. Find the coordinates of the point on the parabola $y^2 = 4ax$ at which the radius of curvature is equal to the latus rectum. (B.A. Sub. 37)

12. Find the radius of curvature at the origin on the curve $x^3 + y^3 + 2x^2 - 4y + 3x = 0$. (B.Sc. 51 T.U.)

13. For the curves $x^2 = 4y$ and $y^2 = 4x$, find (1) the angle of intersection, (2) the sum of curvatures at the point $(4, 4)$. (B.A. 41 M)

14. Show that at any point P on the rectangular hyperbola $xy = c^2$, $\rho = \dfrac{r^3}{2c^2}$ where r is the distance of P from the origin.

15. Find the radius of curvature at the point 'θ' on the curve $x = a \log \sec \theta$; $y = a(\tan \theta - \theta)$. (B.Sc 52 Os. U.)

16. Prove that the radius of curvature of the catenary of uniform strength $y = a \log \sec\left(\dfrac{x}{a}\right)$ is $a \sec \dfrac{x}{a}$. (B.Sc. 52 M)

17. Show that at any point P on the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, the radius of curvature is twice the normal at that point. (B.Sc. 41 M)

18. In the curve $x^3 = y(x-a)^2$, prove that the radius of curvature at the point where the ordinate is minimum is $8a/9$. (B.A. 52 M)

19. Find the point on the curve $y = e^x$ at which the curvature is a maximum and show that the tangent at this point

forms with the axes of coordinates a triangle whose sides are in the ratio of $1 : \sqrt{2} : \sqrt{3}$. (B.Sc. 39 ; B.Sc. 53 T.U.)

20. Find the coordinates of the real points on the curve $y^2 = 2x(3 - x^2)$ the tangents at which are parallel to the x-axis and show that the radius of curvature at each of these points is $\frac{1}{3}$. (B.A. 41 M)

21. For the curve $x = 6t^2 - 3t^4$; $y = 8t^3$, show that the radius of curvature at the point 't' is $6t(1 + t^2)^2$. (B.A. 46 M)

22. Show that the curve represented by the equation $y^3 = x(x + 2y)$ has a minimum ordinate when $x = 1$ and find the radius of curvature at this point. (B.A. 54 M)

23. Show that the projection on the y-axis of the radius of curvature at any point (x, y) of the curve $y = \log \cos x$ is constant. (B.Sc. Sub. 54)

24. If the coordinates of any point of a curve are given by $x = \log t$, $y = (t + t^{-1})/2$, prove that $\frac{d^2y}{dx^2} = y$ and find the radius of curvature at any point (x, y) of the curve. (T.U. 55)

25. The tangents at two points P, Q on the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ are at right angles. Show that if ρ and ρ' be the radii of curvature at these points, $\rho^2 + \rho'^2 = 16a^2$. (B.Sc. 65 M)

§ 40·4. The coordinates of the centre of curvature.

Let the centre of curvature of the curve $y = f(x)$ corresponding to the point $P(x, y)$ be X and Y.

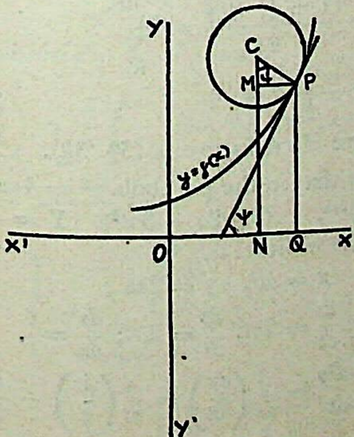

Fig. 32

$$X = ON$$
$$= OQ - NQ = OQ - MP$$
$$= x - PC \sin\Psi = x - \rho \sin\Psi.$$

$$Y = NC = NM + MC$$
$$= QP + PC \cos \Psi = y + \rho \cos \Psi.$$

If y_1 and y_2 denote $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$,

we know that $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$ and $\tan \Psi = y_1$.

$$\therefore \cos \Psi = \frac{1}{\sqrt{1+y_1^2}} \text{ and } \sin \Psi = \frac{y_1}{\sqrt{1+y_1^2}}.$$

$$\therefore X = x - \frac{(1+y_1^2)^{3/2}}{y_2} \cdot \frac{y_1}{(1+y_1^2)^{1/2}} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$Y = y + \frac{(1+y_1^2)^{3/2}}{y_2} \cdot \frac{1}{(1+y_1^2)^{1/2}} = y + \frac{1+y_1^2}{y_2}.$$

The locus of the centre of curvature for a curve is called the *evolute* of the curve.

Examples.

Ex. 1. Find the coordinates of the centre of curvature of the curve $xy = 2$ at the point (2, 1).

Here $y = \frac{2}{x}$. $\therefore \frac{dy}{dx} = -\frac{2}{x^2}$ and $\frac{d^2y}{dx^2} = \frac{4}{x^3}$.

$\therefore$ At (2, 1) the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are respectively $-\frac{1}{2}$ and $\frac{1}{2}$.

$$\therefore \quad X = 2 + \frac{(1+\frac{1}{4}) \times \frac{1}{2}}{\frac{1}{2}} = 3\tfrac{1}{4}.$$

$$Y = 1 + \frac{1+\frac{1}{4}}{\frac{1}{2}} = 3\tfrac{1}{2}.$$

$\therefore$ The centre of curvature is $(3\frac{1}{4}, 3\frac{1}{2})$.

Ex. 2. Show that in the parabola $y^2 = 4ax$ at the point 't', $\rho = -2a(1+t^2)^{3/2}$, $X = 2a + 3at^2$, $Y = -2at^3$. Deduce the equation of the evolute.

$$x = at^2, \quad y = 2at.$$

$$\therefore \frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a \; ; \; \frac{dy}{dx} = \frac{1}{t}.$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{1}{t}\right) \div \frac{dx}{dt}$$

$$= -\frac{1}{t^2} \div 2at = -\frac{1}{2at^3}.$$

$$\therefore \rho = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} \div \frac{d^2y}{dx^2} = -2a(1+t^2)^{3/2}.$$

$$X = x - \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{\frac{d^2y}{dx^2}} = at^2 - \frac{\left(1+\frac{1}{t^2}\right)\frac{1}{t}}{-\frac{1}{2at^3}}$$

$$= 2a + 3at^2.$$

$$Y = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = 2at + \frac{1+\frac{1}{t^2}}{-\frac{1}{2at^2}} = -2at^3.$$

Eliminating t from X and Y,

$$Y = -2a\left(\frac{X-2a}{3a}\right)^{3/2}.$$

Squaring both sides and simplifying, we get

$$27aY^2 = 4(X-2a)^3.$$

The locus of (X, Y) is $27ay^2 = 4(x-2a)^3$.

This curve is called a semi-cubical parabola.

Ex. 3. Find the evolute of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Any point on the ellipse is $(a\cos\theta, b\sin\theta)$.

$$x = a\cos\theta\,;\ \frac{dx}{d\theta} = -a\sin\theta.$$

$$y = b\sin\theta\,;\ \frac{dy}{d\theta} = b\cos\theta.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{b}{a}\cot\theta\right) = \frac{b}{a}\operatorname{cosec}^2\theta\cdot\frac{d\theta}{dx}$$

$$= -\frac{b}{a^2}\operatorname{cosec}^3\theta.$$

$$X = x - \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{\frac{d^2y}{dx^2}}$$

$$= a\cos\theta - \frac{\left(1+\frac{b^2}{a^2}\cot^2\theta\right)\left(\frac{b}{a}\cot\theta\right)}{\frac{b}{a^2}\operatorname{cosec}^3\theta}$$

$$= \frac{(a^2-b^2)\cos^3\theta}{a}.$$

$$Y = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

$$= b \tan \theta - \frac{1 + \frac{b^2}{a^2} \cot^2 \theta}{\frac{b}{a^2} \operatorname{cosec}^3 \theta}$$

$$= -\frac{a^2 - b^2}{b} \sin^3 \theta.$$

$$\cos \theta = \left(\frac{aX}{a^2 - b^2}\right)^{1/3}; \sin \theta = \left(\frac{-bY}{a^2 - b^2}\right)^{1/3}.$$

To eliminate θ, squaring and adding, we get

$$\left(\frac{aX}{a^2 - b^2}\right)^{2/3} + \left(\frac{-bY}{a^2 - b^2}\right)^{2/3} = 1$$

i.e., $$\left(\frac{aX}{a^2 - b^2}\right)^{2/3} + \left(\frac{bY}{a^2 - b^2}\right)^{2/3} = 1.$$

$\therefore$ The locus of (X, Y) is the four cusped hypocycloid

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}.$$

Ex. 4. Show that the evolute of the cycloid $x = a(\theta - \sin \theta)$; $y = a(1 - \cos \theta)$ is another cycloid.

$$\frac{dx}{d\theta} = a(1 - \cos \theta), \frac{dy}{d\theta} = a \sin \theta;$$

$$\therefore \frac{dy}{dx} = \frac{a \sin \theta}{a(1 - \cos \theta)} = \cot \frac{\theta}{2}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\cot \frac{\theta}{2}\right) = -\frac{1}{2} \operatorname{cosec}^2 \frac{\theta}{2} \cdot \frac{d\theta}{dx}$$

$$= -\frac{1}{4a \sin^4 \theta/2}.$$

$$X = x + \frac{(1 + \cot^2 \theta/2) \cot \theta/2}{\frac{1}{4a \sin^4 \theta/2}}$$

$$= a(\theta - \sin \theta) + 2a \sin \theta$$

$$= a(\theta + \sin \theta).$$

$$Y = y + \frac{1 + \cot^2 \theta/2}{-\frac{1}{4a \sin^4 \theta/2}}$$

$$= a(1 - \cos \theta) - 2a(1 - \cos \theta)$$

$$= -a(1 - \cos \theta).$$

$\therefore$ The locus of (X, Y) is

$$x = a\,(\theta + \sin\theta)\ ;\ y = -a\,(1 - \cos\theta).$$

This is also a cycloid.

Exercises XLII.

1. Find the coordinates of the centres of curvature at the given points on the curves :

(1) $y = x^2$; $(\frac{1}{2}, \frac{1}{4})$.

(2) $xy = c^2$; (c, c).

(3) $y = \log\sec x$; $\left(\frac{\pi}{3}, \log 2\right)$.

2. Prove that the circle of curvature at the point $(t^2, 2t)$ of the curve $y^2 = 4x$ cuts the curve again at a point whose ordinate is $-6t$. Calculate the coordinates of the centre of curvature. (B.Sc. 47 M)

3. Show that the evolute of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } (ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}.$$

4. Show that for the curve $x^{2/3} + y^{2/3} = a^{2/3}$ $X = a\cos^3 t + 3a\cos t\sin^2 t$, $Y = a\sin^3 t + 3a\sin t\cos^2 t$.

5. Show that the equation of the evolute of the curve $2xy = a^2$ is $(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$.

6. Prove that if the centre of curvature of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at one end of the minor axis lies at the other end, then the eccentricity of the ellipse is $1/\sqrt{2}$.

7. Show that in the parabola

$$\sqrt{x} + \sqrt{y} = \sqrt{a},\ X + Y = 3\,(x + y).$$

8. Given the curve

$$x = 3\cos t + \cos 3t,\ y = 3\sin t - \sin 3t,$$

find the parametric equations of the evolute. Find the centre of curvature for $t = 0$ and show that it coincides with the corresponding point on the curve.

9. Show that the equation of the circle of curvature at the origin of the parabola $y = mx + \frac{x^2}{a}$ is

$$x^2 + y^2 = (1 + m^2)\,a\,(y - mx). \qquad \text{(B.Sc. 51 T.U.)}$$

§ 40·5. Radius of curvature when the curve is given in polar coordinates.

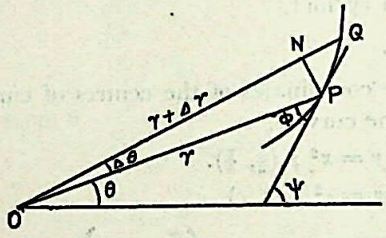

Fig. 33

Let us assume that the equation of the curve in polar coordinates is $r = f(\theta)$.

In the figure, $\Psi = \theta + \phi$.

$$\therefore \frac{d\Psi}{d\theta} = 1 + \frac{d\phi}{d\theta}.$$

We have proved that $\tan \phi = r\dfrac{d\theta}{dr} = \dfrac{r}{\left(\dfrac{dr}{d\theta}\right)}$.

Differentiating w.r.t. θ, we get

$$\sec^2 \phi \frac{d\theta}{d\phi} = \frac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2}.$$

$$\therefore \frac{d\phi}{d\theta} = \frac{1}{\sec^2 \phi} \cdot \frac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2}$$

$$= \frac{1}{1 + \dfrac{r^2}{\left(\dfrac{dr}{d\theta}\right)^2}} \cdot \frac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2}$$

$$= \frac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^3}}{\left(\dfrac{dr}{d\theta}\right)^2 + r^2}$$

$$\frac{d\Psi}{d\theta} = 1 + \frac{d\phi}{d\theta}$$

$$= 1 + \frac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2 + r^2}$$

$$= \frac{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2 + r^2}.$$

We have proved in the previous chapter that

$$\frac{ds}{d\theta} = \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{1/2}.$$

$$\rho = \frac{ds}{d\Psi} = \frac{ds}{d\theta} \cdot \frac{d\theta}{d\Psi}$$

$$= \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{1/2} \cdot \frac{\left(\dfrac{dr}{d\theta}\right)^2 + r^2}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}$$

$$= \frac{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}.$$

Examples.

Ex. 1. Find the radius of curvature of the cardioid

$$r = a(1 - \cos\theta).$$

Here $\dfrac{dr}{d\theta} = a\sin\theta$, $\dfrac{d^2r}{d\theta^2} = a\cos\theta$.

$$\therefore \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2} = \{a^2(1-\cos\theta)^2 + a^2\sin^2\theta\}^{3/2}$$

$$= 8a^3\sin^3\frac{\theta}{2}.$$

$$r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2} =$$

$$a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a^2\cos\theta(1-\cos\theta)$$

$$= 6a^2\sin^2\theta/2.$$

$$\therefore \rho = \frac{8a^3\sin^3\theta/2}{6a^2\sin^2\theta/2} = \tfrac{4}{3}a\sin\theta/2$$

$$= \tfrac{2}{3}\sqrt{2ar}.$$

Ex. 2. Show that the radius of curvature of the curve

$$r^n = a^n \cos n\theta \text{ is } \frac{a^n r^{-n+1}}{n+1}.$$

Taking logarithms on both sides and differentiating, we get

$$\frac{n}{r}\frac{dr}{d\theta} = -\frac{n \sin n\theta}{\cos n\theta}.$$

$$\therefore \frac{dr}{d\theta} = -r \tan n\theta.$$

Differentiating once again w.r.t. θ, we get

$$\frac{d^2r}{d\theta^2} = -\frac{dr}{d\theta} \tan n\theta - nr \sec^2 n\theta$$

$$= r \tan^2 n\theta - nr \sec^2 n\theta.$$

$$\rho = \frac{(r^2 + r^2 \tan^2 n\theta)^{3/2}}{r^2 + 2r^2 \tan^2 n\theta - r^2 \tan^2 n\theta + nr^2 \sec^2 n\theta}$$

$$= \frac{r^3 \sec^3 n\theta}{(n+1) r^2 \sec^2 n\theta} = \frac{r}{(n+1) \cos n\theta}$$

$$= \frac{r \cdot a^n}{(n+1) r^n} = \frac{a^n r^{-n+1}}{n+1}.$$

Particular cases.

(*i*) Putting $n = 2$, we get Bernouilli's lemniscate; $\rho = a^2/3r$.

(*ii*) When $n = -2$, we have a rectangular hyperbola; $\rho = r^3/a^2$.

(*iii*) When $n = \frac{1}{2}$, we get a cardioid ; $\rho = \frac{2}{3}\sqrt{ar}$.

(*iv*) When $n = -\frac{1}{2}$, we get a parabola ; $\rho = 2r^{3/2}/\sqrt{a}$.

(*v*) When $n = 1$, we get a circle ; $\rho = \frac{a}{2}$.

§ 40·6. p-r equation of curve.

Let OA be the *initial* line, O the pole and P any point on the curve. The length of the perpendicular drawn from

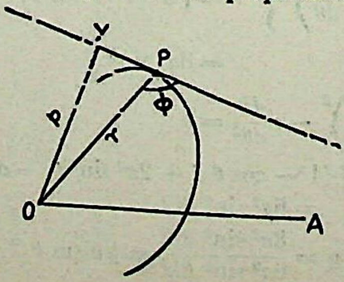

Fig. 34

the pole on the tangent at the point P is usually denoted by p.

$$OY = p = r \sin O\hat{P}Y = r \sin (180 - \phi) = r \sin \phi.$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2 \sin^2 \phi}$$

$$= \frac{1}{r^2}(1 + \cot^2 \phi)$$

$$= \frac{1}{r^2}\left\{1 + \left(\frac{1}{r}\frac{dr}{d\theta}\right)^2\right\}.$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

If $r = \dfrac{1}{u}, \dfrac{dr}{d\theta} = -\dfrac{1}{u^2}\dfrac{du}{d\theta}$.

$$\therefore \quad \frac{1}{p^2} = u^2 + u^4\left(-\frac{1}{u^2}\frac{du}{d\theta}\right)^2$$

$$= u^2 + \left(\frac{du}{d\theta}\right)^2.$$

Examples.

Ex. 1. Prove that the (p, r) equation of the cardioid $r = a(1 - \cos\theta)$ is $p^2 = \dfrac{r^3}{2a}$.

$$r = a(1 - \cos\theta). \quad \therefore \quad \frac{dr}{d\theta} = a \sin\theta.$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} + \frac{1}{r^4}a^2\sin^2\theta$$

$$= \frac{r^2 + a^2\sin^2\theta}{r^4} = \frac{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}{r^4}$$

$$= \frac{2a^2 - 2a^2\cos\theta}{r^4} = \frac{2a^2(1-\cos\theta)}{r^4}.$$

$$= \frac{2ar}{r^4} = \frac{2a}{r^3}.$$

$$p^2 = r^3/2a.$$

Ex. 2. From the polar equation of the parabola, show that $p^2 = ar$.

Polar equation of the parabola is $\dfrac{2a}{r} = 1 - \cos\theta$ (with respect to the focus as pole).

Differentiating both sides with respect to θ, we get

$$-\frac{2a}{r^2}\frac{dr}{d\theta}=\sin\theta.$$

$$\therefore \frac{1}{r^2}\frac{dr}{d\theta}=-\frac{\sin\theta}{2a}.$$

$$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{1}{r^2}+\frac{\sin^2\theta}{4a^2}$$

$$=\frac{(1-\cos\theta)^2}{4a^2}+\frac{\sin^2\theta}{4a^2}=\frac{2-2\cos\theta}{4a^2}$$

$$=\frac{2a}{r}\cdot\frac{2}{4a^2}=\frac{1}{ar}.$$

$$\therefore p^2=ar.$$

§ 40·7. For some curves it is not easy to calculate the radius of curvature from their polar coordinates. In those cases, we can use the following formula.

We have proved that

$$\sin\phi=r\frac{d\theta}{ds};\ \cos\phi=\frac{dr}{ds};\ \tan\phi=r\frac{d\theta}{dr}.$$

$$p=r\sin\phi.$$

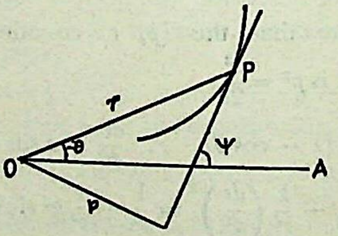

Fig. 35

$$\frac{dp}{dr}=\sin\phi+r\cos\phi\frac{d\phi}{dr}=r\frac{d\theta}{dr}+r\frac{dr}{ds}\frac{d\phi}{dr}$$

$$=r\frac{d\theta}{ds}+r\frac{d\phi}{ds}=r\left(\frac{d\theta}{ds}+\frac{d\phi}{ds}\right)$$

$$=r\frac{d}{ds}(\theta+\phi)=r\frac{d\Psi}{ds}.$$

$$\frac{ds}{d\Psi}=r\frac{dr}{dp}.\quad\therefore\ \rho=r\frac{dr}{dp}.$$

Examples.

Ex. 1. Find the radius of curvature of the cardioid $r=a(1-\cos\theta)$.

We have shown that the (p, r) equation of the cardioid is $2p^2a = r^3$.

Differentiating with respect to p on both sides, we get

$$4pa = 3r^2 \frac{dr}{dp}.$$

$$r\frac{dr}{dp} = \frac{4pa}{3r}.$$

$$\therefore \rho = \frac{4pa}{3r} = \frac{4a}{3r}\cdot\left(\frac{r^3}{2a}\right)^{1/2} = \frac{2}{3}(2r)^{1/2}.$$

Ex. 2. Find the radius of curvature of the curve $r^2 = a^2 \sin 2\theta$.

Differentiating with respect to θ on both sides, we get

$$2r\frac{dr}{d\theta} = 2a^2 \cos 2\theta.$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4}\,\frac{a^4\cos^2 2\theta}{r^2} = \frac{r^4 + a^4\cos^2 2\theta}{r^6}$$

$$= \frac{a^4\sin^2 2\theta + a^4\cos^2 2\theta}{r^6} = \frac{a^4}{r^6}.$$

$$\therefore \; p = \frac{r^3}{a^2}.$$

Differentiating with respect to r, we get

$$\frac{dp}{dr} = \frac{3r^2}{a^2}$$

$$\rho = r\frac{dr}{dp} = r\,\frac{a^2}{3r^2} = \frac{a^2}{3r}.$$

Exercises XLIII.

1. Find the radius of curvature for the following curves :—
   (1) rectangular hyperbola $r^2 = a^2 \sec 2\theta$.
   (2) conic $\frac{l}{r} = 1 + e\cos\theta$.
   (3) parabola $r\cos^2\theta/2 = a$.
   (4) $r = a\theta$.
   (5) $r = Ke^{\theta\cot\alpha}$, where K and α are constants.

2. Show that the radius of curvature of the curve $r^2 = a^2\cos 2\theta$ is $a^2/3r$.

3. Find the radius of curvature at (r, θ) on the curve $r^n = a^n \sin n\theta$.

4. Show that in the cardioid $r = a(1 + \cos\theta)$

$\frac{\rho^2}{r}$ is constant.

5. Show that the curvatures of the curves $r = a\theta$ and $r\theta = a$ at their intersecting point are in the ratio $3 : 1$.

6. For the curve $r = a \sec 2\theta$, show that $\rho = -\frac{r^4}{3p^3}$ where p is the length of the perpendicular from the pole on the tangent at the point (r, θ).

7. Show that the radius of curvature of the curve $r = a \sin n\theta$ at the pole is $na/2$.

8. Find the $(p\text{-}r)$ equations of the following curves :—

(1) $r \sin\theta + a = 0$.

(2) $r = \frac{a}{2}(1 - \cos\theta)$.

(3) $r = a \sin\theta$.

(4) $r^2 \sin 2\theta + a^2 = 0$.

(5) $r^m = a^m \sin m\theta$.

(6) $r = a\, e^{\theta \cot \alpha}$.

(7) $r = a \sin^3 \frac{\theta}{3}$.

9. In any curve, prove that

(i) $p = r^2 \frac{d\theta}{ds}$, (ii) $\sqrt{r^2 - p^2} = r\frac{dr}{ds}$.

CHAPTER XI

TRACING OF CURVES

§ 41·1. Elementary method of tracing curves is to plot down a number of points whose co-ordinates satisfy the equation of the curve and to draw a smooth curve through them. If the equation of the curve is given in the form $f(x, y) = 0$, solve for y or x; assume arbitrary values for x (or y) and calculate the corresponding values for y (or x) and thus the co-ordinates of a number of points on the curve are known.

Take the curve $y = x^2 (x - 1)$.

x	0	1	2	3	4	5	-1	-2	-3	-4	-5
$y = x^2 (x-1)$	0	0	4	18	48	100	-2	-12	-36	-80	-150

Thus the points (0, 0), (1, 0), (2, 4), (3, 18), (4, 48), (5, 100), $(-1, -2)$, $(-2, -12)$, $(-3, -36)$, $(-4, -80)$, $(-5, -150)$ lie on the curve. We can plot these points on a graph paper and get the curve in Fig. 36.

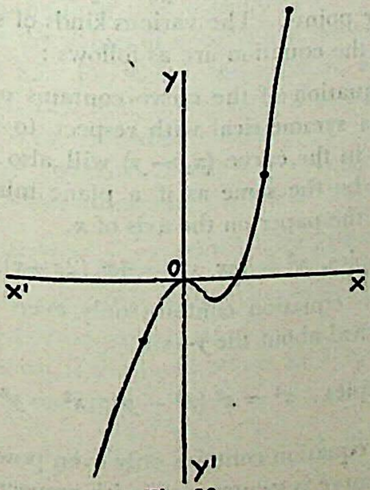

Fig. 36

We are doubtful about the shape of the curve between the points (0, 0) and (1, 0). So we get more points on the curve between these points. Put $x = \frac{1}{2}$; then $y = -\frac{1}{8}$.

Put $x = \frac{1}{3}$; then $y = -\frac{2}{27}$.

$\therefore$ The points $\left(\frac{1}{2}, -\frac{1}{8}\right)$, $\left(\frac{1}{3}, -\frac{2}{27}\right)$ lie on the curve.

Plot these points and we can roughly see the shape of the curve.

The process is laborious and in case the equation of the curve is of a degree higher than the second, for example, $x^4 - 3axy^2 + 2ay^3 = 0$, it is not possible to solve the equation for x or y and so we cannot find easily the number of points on the curves. From the equations of the curve, we can deduce many properties of the curve and from those properties we can draw the curve without plotting many points. Moreover the general form of a curve is usually all that is desired and Calculus furnishes us with powerful methods for determining the shape of a curve with very little computation.

§ 41·2. To trace the graph of a curve whose equation is given in Cartesian co-ordinates, it is better to adopt a method as detailed below.

The first consideration is the *symmetry* of a curve with respect to certain lines or points. The various kinds of symmetry arising from the form of the equation are as follows :

(1) If the equation of the curve contains only even powers of y, the curve is symmetrical with respect to the x-axis, for if (x, y) be a point in the curve $(x, -y)$ will also be on the curve. The figure will be the same as if a plane mirror were placed perpendicular to the paper on the axis of x.

Examples. $y^2 = 4ax$, $xy^2 = 4a^2(2a - x)$, $y^2(2a - x) =$

(*a*) If the equation contains only even powers of x, the curve is symmetrical about the y-axis.

Examples. $x^4 = a^2(x^2 - y)$; $x^2 = y^2 \frac{y + a}{y - a}$.

If the equation contains only even powers of x and even powers of y, the curve is symmetrical with respect to both axes.

Examples. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(x^2 - a^2)(y^2 - b^2) =$

$x^2y^2 = a^2(x^2 - y^2)$.

(*b*) If the equation be not altered when $-x$ and $-y$ are written for x and y, the curve is symmetrical with respect to the origin in opposite quadrants.

Examples. $xy = c^2$; $x^4 = a^2(x^2 - y^2)$, $x^3 + y^3 = a^2 x$.

(*c*) If the equation be not altered when x and y are interchanged, the curve is symmetrical with respect to a line bisecting $X\hat{O}Y$, *i.e.*, $y = x$.

Example. $xy = c^2$, $x^3 + y^3 = 3axy$.

(2) ***Special points*** on the curve.

(*a*) Find the points where the curve crosses the x-axis. To find those points, substitute 0 for y in the curve and solve the resulting equation in x. Similarly find the points where the curve crosses the y-axis.

(*b*) If the equation of the curve does not involve the constant term, it passes through the origin.

(*c*) Find the values of x which will make the values of y imaginary. Let those values be $x = \alpha$ and $x = \beta$. Then no real part of the curve will lie between the lines $x = \alpha$ and $x = \beta$. Similarly find the values of y which will make the values of x imaginary. If those values of y are γ and δ, the curve will not lie between the lines $y = \gamma$ and $y = \delta$.

(*d*) Find the values of x for which the value of y is infinity and the values of y for which the value of x is infinity. From these values, we can find the equations to the asymptotes of the curve parallel to the co-ordinate axes.

(*e*) If the equation of the curve is given as an implicit equation, it is better to express the curve in the form $y = f(x)$.

For example, $y = \dfrac{x^2}{1 + x^2}$ can be put in the form $x^2 = \dfrac{y}{1 - y}$; when $y \to 1$, the value of x approaches infinity.

$\therefore$ $y = 1$ is an asymptote.

When $x \to \infty$, if the limit of y is $ax + b$, then $y = ax + b$ is an asymptote to the curve.

Example. In $y = \dfrac{1}{2}x + 3 + \dfrac{1}{x}$ as x tends to infinity, the limit of y is $\dfrac{1}{2}x + 3$.

$\therefore$ $y = \dfrac{1}{2}x + 3$ is an asymptote.

(3) (*a*) From the equation of the curve find the value of $\frac{dy}{dx}$. The value of $\frac{dy}{dx}$ at a point gives the slope of the curve at that point.

(*b*) Find $\frac{d^2y}{dx^2}$. From this we can determine the range within which the curve is concave upwards or concave downwards and the points of inflexion.

(*c*) Find the points where the curve attains its maximum or minimum if any.

Examples.

Ex. 1. Trace the curve whose equation is

$$y = \frac{x^2+1}{x^2-1}.$$

(1) *Symmetry.* Since the terms involving x are even powers, the curve is symmetrical about the y-axis.

(2) *Special points.* (*a*) $y = -1$ when $x = 0$; when $y = 0$, x has imaginary values.

$\therefore$ The curve will not intersect the x-axis but will cross the y-axis at the point $(0, -1)$.

(*b*) We can write the equation of the curve as

$$x^2 = \frac{y+1}{y-1}.$$

When the value of y lies between $+1$ and -1, the value of x^2 is negative.

$\therefore$ For those values of y, x is imaginary.

$\therefore$ The curve does not lie between the lines $y = 1$ and $y = -1$.

(*c*) y tends to infinity when x tends to $+1$ or to -1.

$\therefore$ $x = 1$ and $x = -1$ are asymptotes of the curve.

x tends to infinity when y tends to 1.

$\therefore$ $y = 1$ is an asymptote.

(3) $y = \dfrac{x^2+1}{x^2-1}$

$$\frac{dy}{dx} = -\frac{4x}{(x^2-1)^2}, \quad \frac{d^2y}{dx^2} = \frac{4(x^2+1)}{(x^2-1)^3}.$$

$$\frac{dy}{dx} = 0 \text{ when } x = 0.$$

$\therefore$ The curve attains its maximum when $x = 0$ and the maximum value is -1.

(4) Giving different values for x, calculate the corresponding values for y and tabulate the values as below :

x	1	2	3	4	5	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
y	∞	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{17}{15}$	$\frac{13}{12}$	$-\frac{17}{15}$	$-\frac{5}{3}$	$-\frac{25}{7}$

The shape of the curve known from these is as below :

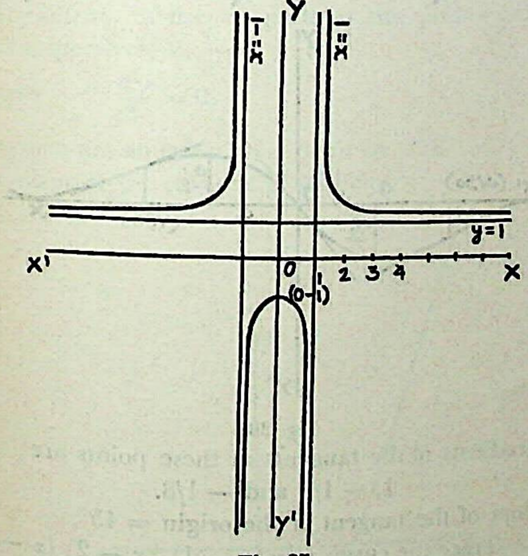

Fig. 37

Ex. 2. Trace the curve $(a^2+x^2)y = a^2x$.

Symmetry. The equation of the curve is unaltered after substituting $-x$ and $-y$ for x and y. Hence the curve is symmetrical in the opposite quadrants.

Special points. The curve passes through the origin

$$y = \frac{a^2x}{a^2+x^2}.$$

As x increases, the value of y decreases and when x tends to $\pm\infty$, y tends to 0.

$\therefore$ $y = 0$ is an asymptote to the curve.

$$\frac{dy}{dx}=\frac{a^2(a^2-x^2)}{(a^2+x^2)^2}.$$

$$\frac{d^2y}{dx^2}=-\frac{2a^2x(3a^2-x^2)}{(a^2+x^2)^3}.$$

$\therefore$ y attains its maximum at $x=a$, and its minimum at $x=-a$.

Its maximum value $=a/2$.

Its minimum value $=-a/2$.

Its inflexional points are

$$(0,0),\ \left(\sqrt{3}a, \frac{\sqrt{3}}{4}a\right),\ \left(-\sqrt{3}a, \frac{-\sqrt{3}}{4}a\right).$$

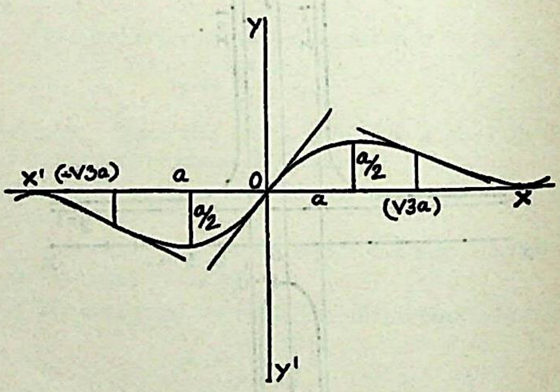

Fig. 38

The gradients of the tangents at these points are

$$1,\ -1/8 \text{ and } -1/8.$$

The slope of the tangent at the origin $=45°$.

Ex. 3. Trace the curve $y=(x-1)(x-2)(x-3)$.

There is no symmetry in the curve.

Special points. The curve crosses the x-axis at 1, 2 and 3 and crosses the y-axis at -6.

As the value of x increases, the value of y increases, for $x>3$.

When x tends to infinity, y also tends to ∞.

The value of y is positive when x lies between 1 and 3

,, is negative ,, ,, 2 and 3

,, is negative when $x<1$

When x tends to $-\infty$, y also tends to $-\infty$.

$$y=x^3-6x^2+11x-6.$$

$$\frac{dy}{dx}=3x^2-12x+11.$$

When $3x^2 - 12x + 11 = 0$, $x = \dfrac{12 \pm \sqrt{144 - 132}}{6}$.

$\therefore\ x = 1{\cdot}4$ or $2{\cdot}6$ approximately.

$$\frac{d^2y}{dx^2} = 6x - 12.$$

$\therefore$ The curve attains its maximum at $x = 1{\cdot}4$ and its minimum at $x = 2{\cdot}6$.

Its maximum value $= {\cdot}384$.

Its minimum value $= -{\cdot}384$.

The gradients of the tangents at the points $x = 1$, $x = 2$, $x = 3$ are respectively $2, -1, +1$.

When $x = 2$, $\dfrac{d^2y}{dx^2} = 0$.

The curve has an inflexional point at $(2, 0)$.

The shape of the curve is given below :

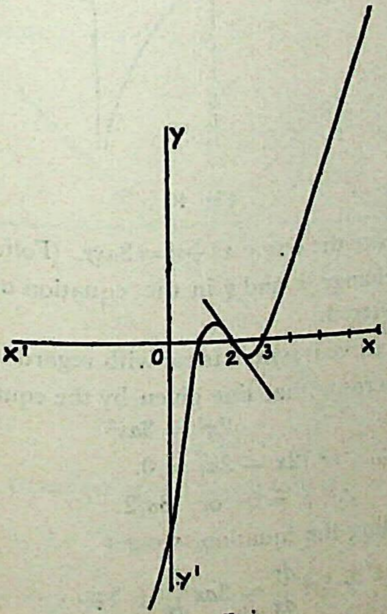

Fig. 39

Ex. 4. Trace the curve $y^2 = x^2 \dfrac{a + x}{b - x}$.

Symmetry. Since the equation involves only even powers of y, the curve is symmetrical with regard to the x-axis.

The curve crosses the x-axis at $x = 0$ and $x = -a$ and the[n]
forms a loop.

If the value of x is greater than b but less than $-a$, then y^2 is
negative.

$\therefore$ For those values of x, y is imaginary. So the curve does
not lie in the plane where $x > b$ and $x < -a$.

$$y \to \infty \text{ when } x = b.$$

$\therefore$ $x = b$ is an asymptote to the curve.

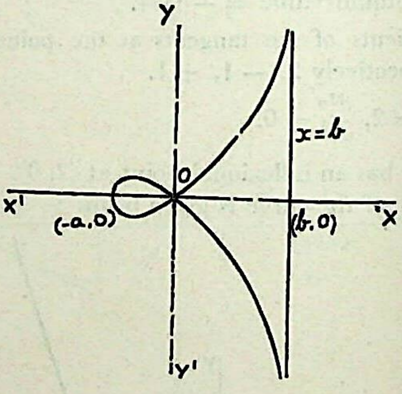

Fig. 40

Ex. 5. Trace the curve $x^3 + y^3 = 3axy$. (Folium of Descartes)

If we interchange x and y in the equation of the curve, th[e]
equation is unaltered.

$\therefore$ The curve is symmetrical with regard to the line $y = x$.

The curve crosses that line given by the equation

$$2x^3 = 3ax^2$$

$$\text{i.e.,} \quad x^2(2x - 3a) = 0.$$

$$\therefore \quad x = 0 \quad \text{or} \quad 3a/2.$$

Differentiating the equation, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 3ax\frac{dy}{dx} + 3ay.$$

$$\therefore \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}.$$

At the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, the value of $\frac{dy}{dx} = -1$.

$\therefore$ At this point, the tangent to the curve makes an angle of $135°$ with the x-axis.

When $x = 0$, y has got two zero values.

$\therefore$ $x = 0$ is a tangent to the curve.

Similarly $y = 0$ is a tangent to the curve.

$$x^3 + y^3 = 3axy.$$

$$y^3 = 3axy - x^3$$

$$= -x^3\left(1 - \frac{3ay}{x^2}\right).$$

$$\therefore \quad y = -x\left(1 - \frac{3ay}{x^2}\right)^{1/3}$$

$$= -x\left(1 - \frac{ay}{x^2} + \ldots\right)$$

$$= -x + \frac{ay}{x} + \ldots$$

The asymptote is parallel to the line $y = -x$.

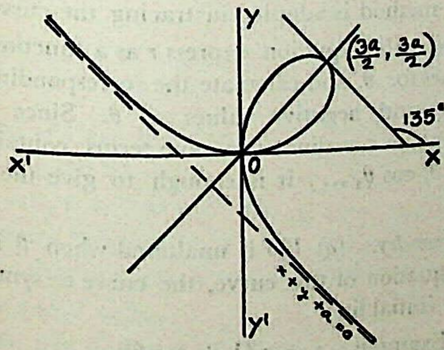

Fig. 41

$\therefore$ The equation of the asymptote is

$$y + x = \underset{y \to -x}{\text{Lt}} \frac{ay}{x} = -a$$

i.e., $\quad x + y + a = 0.$

Exercises XLIV.

Trace the curves :

1. $y^2 = (x-1)(x-2)^2$. 2. $y = \dfrac{x}{(2-x)^2}$.

3. $y^2 = \dfrac{x^2(4-x)}{3+x}$. 4. $y^2 = \dfrac{x}{3-x}$.

5. $y^2(a+x) = x^2(3a-x)$. 6. $ay^2 = a^2(x-a)$.

7. $9ay^2 = x(x-3a)^2$.

8. $y^2(x^2+y^2) + a^2(x^2-y^2) = 0$.

9. $y^2(a+x) = (a-x)x^2$.

10. $y^2(x^2-a^2) = x^2(x^2-4a^2)$.

11. $xy^2 = 4a^2(2a-x)$. 12. $x^2y^2 = a^2(x^2-y^2)$.

13. $y^2 = \dfrac{x^2(2+x)}{2-x}$. 14. $y^2 = x^2(x-3)$.

15. $y = \dfrac{1}{3}x - \dfrac{2}{x}$.

16. $y = e^x$ and $y = e^{-x^2}$. (B.Sc. Comp. 59)

17. $y = \log e^x$. (B.Sc. Anc. 59)

§ 41·3. Polar equation.

If the equation of the curve is given in polar coordinates, the following method is adopted in tracing the curve :—

(1) From the equation express r as a function of θ. Give different values for θ, and calculate the corresponding values for r. Give positive and negative values to θ. Since most of the equations in polar coordinates involve terms containing periodic functions $\sin\theta$, $\cos\theta$, ... , it is enough to give the values for θ from 0 to 2π.

(2) *Symmetry*. (*a*) If r is unaltered when θ is changed to $-\theta$ in the equation of the curve, the curve is symmetrical with regard to the initial line.

Example. $r = a(1+\cos\theta)$.

(*b*) If the equation involves only even powers of r, the curve is symmetrical with regard to the origin. In this case the pole is the centre of the curve.

(3) If the value of $\tan\phi$ is calculated, the slope of the tangent with the radius vector can be found.

(4) Find the value of θ which makes r infinity. The curve has an asymptote in that direction.

(5) In certain equations, r has limits and find those limits.

Example. $r = a\sin n\theta$; limits of r are 0 and a.

Ex. 6. Trace the curve $r = a \sin 3\theta$.

Sin 3θ is a periodic function with a period of 2π.

Give θ different values and calculate the corresponding values of r. Those values are tabulated below :

3θ	0	Intermediate values.	$\frac{\pi}{2}$	Intermediate values.	π	Intermediate values.	$\frac{3\pi}{2}$	Intermediate values.	2π
θ	0		$\frac{\pi}{6}$		$\frac{\pi}{3}$		$\frac{\pi}{2}$		$\frac{2\pi}{3}$
r	0	Increasing +ve values.	a	Decreasing +ve values.	0	Decreasing −ve values.	$-a$	Increasing +ve values.	0

The limits of r are 0 and a.

$\tan \phi = \frac{1}{3} \tan 3\theta$.

When $r = 0$, $\theta = 0$ or π or 2π.

$\therefore$ For these values $\phi = 0$.

$\therefore$ The initial line is a tangent to the curve at the points $\theta = 0$, $\theta = \pi$ and $\theta = 2\pi$.

When $r = a$, $a = a \sin 3\theta$.

$$\therefore \quad 3\theta = \frac{\pi}{2}. \quad \therefore \quad \theta = \frac{\pi}{6}.$$

At this point $\tan \phi = \frac{1}{3} \tan \frac{\pi}{2} = \infty$. $\therefore \phi = \frac{\pi}{2}$.

$\therefore$ At $\left(\frac{\pi}{6}, a\right)$ the tangent is $\perp$ to the radius vector.

Similarly at the points $\left(\frac{5\pi}{6}, a\right)\left(\frac{3\pi}{2}, a\right)$ the tangents are $\perp$ to the radius vectors at those points.

C—15

The shape of the curve is given below :

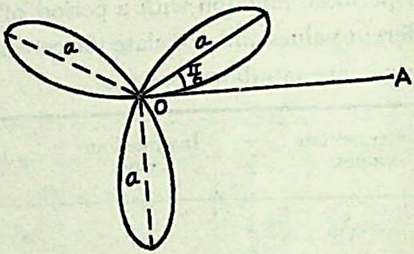

Fig. 42

Ex. 7. Trace the curve $r = a \sin 4\theta$. The values of r corresponding to different values of θ are tabulated below :

4θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
θ	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π
r	0	a	0	$-a$	0	a	0	$-a$	0

$$\tan \phi = \tfrac{1}{4} \tan 4\theta.$$

If the value of r is zero, $\sin 4\theta = 0$.

Then $4\theta = \pm n\pi$.

In that case $\tan \phi = \frac{1}{4} \tan (\pm n\pi) = 0$.

When the value of $r = \pm a$, $\tan \phi = \infty$.

Therefore this curve has 8 similar loops symmetric to the pole and all these loops lie within a circle with the pole as centre and radius a.

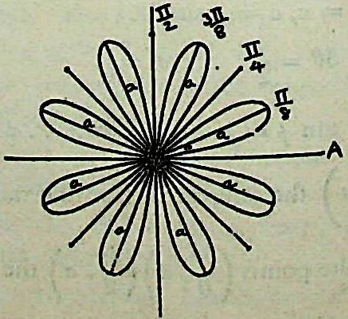

Fig. 43

Ex. 8. Trace the curve $r = a + b \cos \theta$.

Case (*i*). Let $a > b$.

Here let us draw the curve $r = 3 + 2 \cos \theta$.

θ	0	Values from 0 to $\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	5	Decreasing +ve values	4	3	2	1

Changing the sign of θ to $-\theta$, we get

$$r = 3 + 2 \cos(-\theta) = 3 + 2 \cos \theta.$$

There is no change in the equation. Therefore the curve is symmetrical with respect to the initial line.

$$\tan \phi = \frac{r}{\frac{dr}{d\theta}} = \frac{3 + 2 \cos \theta}{-2 \sin \theta}.$$

When $\theta = 0$, $\tan \phi = \infty$. $\therefore \phi = \frac{\pi}{2}$.

When $\theta = \pi$, $\tan \phi = \infty$. $\therefore \phi = \frac{\pi}{2}$.

Tracing the curve, we will get the following curve :

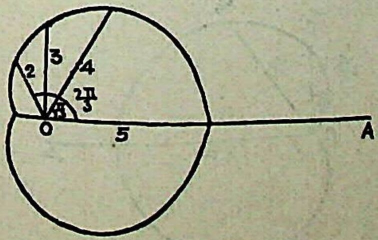

Fig. 44

Case (*ii*). $a < b$. Here let us draw the curve $r = 1 + 2\cos\theta$. This curve is symmetrical with respect to the initial line.

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	Values lying between $2\pi/3$ and π
r	3	2	1	0	Decreasing negative values

$$\tan\phi = \frac{1 + 2\cos\theta}{-2\sin\theta}.$$

When $\theta = 0$ or π, $\tan\phi = \infty$. $\therefore\ \phi = \frac{\pi}{2}$.

$\therefore$ At these points the tangents are $\perp$ to the initial line.

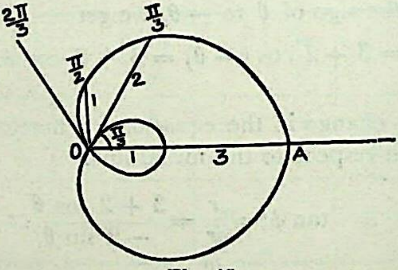

Fig. 45

Case (*iii*). $a = b$.

The curve becomes $r = a(1 + \cos\theta)$. This curve is called a *cardioid*.

This curve is symmetrical with respect to the initial line.

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	$2a$	$\frac{3a}{2}$	a	$\frac{\pi}{2}$	0

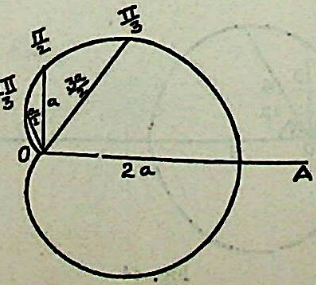

Fig. 46

Ex. 9. Trace the curve $r^2 = a^2 \cos 2\theta$.

(Lemniscate of Bernouilli.)

There is no change in the equation of the curve when r and θ are changed to $-r$ and $-\theta$ respectively. Therefore the curve is symmetrical with regard to the initial line and to the pole.

If $\cos 2\theta$ is negative, r^2 is negative.

$\therefore$ r is imaginary for those values of θ.

$\therefore$ When θ lies between $+\frac{\pi}{4}$ and $\frac{3\pi}{4}$, r is imaginary.

$\therefore$ There is no part of the curve between the lines

$$\theta = \frac{\pi}{4} \text{ and } \theta = \frac{3\pi}{4}.$$

When $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}, r = 0.$

When $\theta = 0, \pi, r = \pm a.$

When the value of θ increases from $-\frac{\pi}{4}$ to 0, the value of r increases from 0 to a. When the value of θ increases from 0 to $\frac{\pi}{4}$, the value of r decreases from a to 0.

The value of r cannot be greater than a. This curve consists of two similar loops.

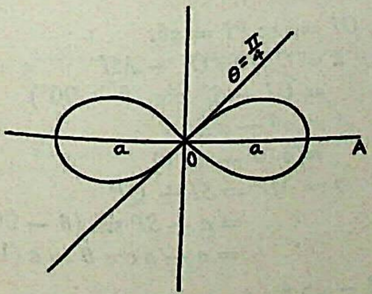

Fig. 47

Exercises XLV.

Trace the following curves :—

(1) $r = a(2\cos\theta + 1)$. (2) $r = a(2\cos\theta - 1)$.
(3) $r = a\theta$. (4) $r\theta = a$. (5) $r = ae^{\theta\cot\alpha}$.

Well-known Curves

(*a*) *Cycloid.* When a circle rolls on a fixed straight line without slipping, the path traced out by a fixed point on the circumference of a circle is called a *cycloid* and the circle is called the *generating circle* of the cycloid.

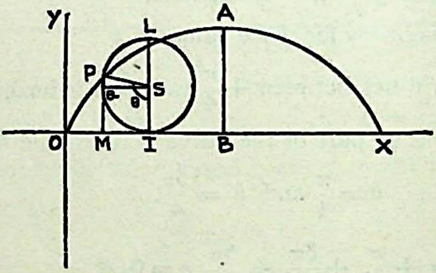

Fig. 48

Let us assume that OX be the base, LPI be the generating circle, the fixed point on the circle be P, the angle between the radii SP and SI be θ and I be the point of contact of the circle with the base OX.

Let us assume also that when the circle begins to roll the position of the point P is at O.

Take OX and the line through $O \perp$ to OX as coordinate axes. Let the coordinates of P be (x, y). Draw $PM \perp$ to OX and $SQ \perp$ to MP.

$$OI = \text{arc } PI = a\theta.$$

$$\begin{aligned} x = OM &= OI - MI \\ &= OI - SP \cos(\theta - 90°) \\ &= a\theta - a \sin\theta \\ &= a(\theta - \sin\theta). \end{aligned}$$

$$\begin{aligned} y = MP &= SI + QP \\ &= a + SP \sin(\theta - 90°) \\ &= a - a\cos\theta = a(1 - \cos\theta). \end{aligned}$$

$$\therefore \quad x = a(\theta - \sin\theta)$$

$y = a(1 - \cos\theta)$ are the equations of the cycloid.

When $\theta = \pi$, $x = a\pi = OB$. Then P is at A.

When $\theta = 2\pi$, $x = 2a\pi = OX$. Then P is at X.

The curve is symmetrical with respect to the line BA.

A is called the *vertex*, and OX the *base* of the cycloid.

When the equations of the curve are given as

$x = a(\theta + \sin\theta)$

$y = a(1 - \cos\theta)$, the vertex is taken as the origin and the coordinate axes are taken as in the following figure :

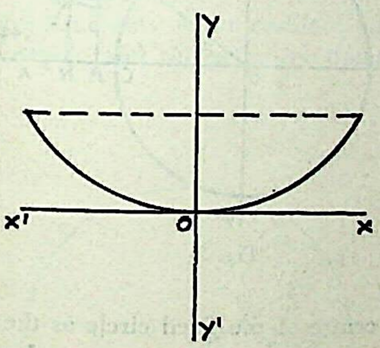

Fig. 49

When the equations of the curve are given as

$x = a(\theta + \sin\theta)$

$y = a(1 + \cos\theta)$, the vertex of the curve is taken as the origin and the axes are taken as in the following figure :

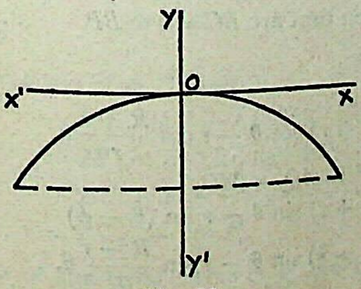

Fig. 50

Epi- and Hypocycloids

(*b*) When a circle rolls without sliding on the circumference of a fixed circle, the path traced by a point on the circumference of the rolling circle, is called an epi- or hypocycloid according

as the moving circle rolls on the exterior or the interior of the other.

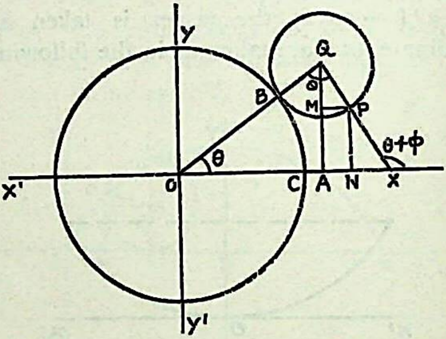

Fig. 51

(*i*) *Epicycloid.*

Assume the centre of the fixed circle as the origin and two perpendicular diameters as coordinate axes. Let the position of the point P when the outer circle started to roll be C and let $\angle COQ$ be θ, $\angle BQP$ be ϕ, the radii of the circles be R and r and the coordinates of P be x and y.

$$\begin{aligned} x &= ON = OA + MP \\ &= OQ \cos\theta + PQ \cos \angle QPM \\ &= (R+r)\cos\theta + r\cos(180^\circ - \overline{\theta+\phi}) \\ &= (R+r)\cos\theta - r\cos(\theta+\phi). \end{aligned}$$

It is easily seen that arc BC = arc BP. $\therefore\ R\theta = r\phi$.

$$\therefore\quad \phi = \frac{R}{r}\theta.$$

$$\therefore\quad x = (R+r)\cos\theta - r\cos\frac{R+r}{r}\theta.$$

$$\begin{aligned} y &= NP = AQ - MQ \\ &= (R+r)\sin\theta - r\sin(\theta+\phi) \\ &= (R+r)\sin\theta - r\sin\frac{R+r}{r}\theta. \end{aligned}$$

(*ii*) *Hypocycloid.*

Changing r to $-r$ in the equation of the epicycloid, we get the equations of the hypocycloid

$$x = (R-r)\cos\theta + r\cos\frac{R-r}{r}\theta.$$

$$y = (R-r)\sin\theta - r\sin\frac{R-r}{r}\theta.$$

From these equations, we can derive the equations of two special curves.

(1) Let $R = r$ in the equation of the epicycloid.

Then $x = R\,(2 \cos \theta - \cos 2\theta)$

$y = R\,(2 \sin \theta - \sin 2\theta)$.

These are the equations of the cardioid. In the epicycloid assuming C as the pole and the polar coordinates of P as (r, θ), we get

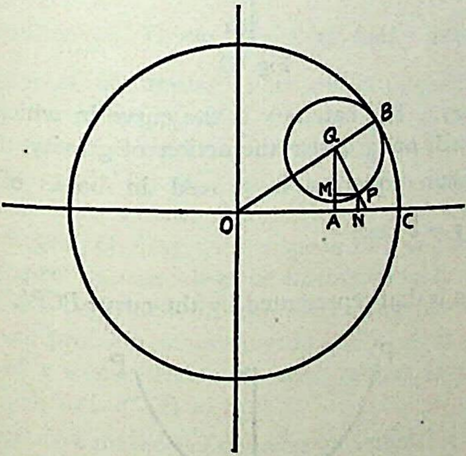

Fig. 52

$$\begin{aligned} r \cos \theta &= CN = OC - CN \\ &= R\,(2 \cos \theta - \cos 2\theta) - R \\ &= 2R \cos \theta - 2R \cos^2 \theta \\ &= 2R \cos \theta\,(1 - \cos \theta) \\ \therefore\ r &= 2R\,(1 - \cos \theta). \end{aligned}$$

If the radius of the fixed circle is a, the polar equation of the cardioid is $r = 2a\,(1 - \cos \theta)$.

(2) Let $R = 4r$ in the hypocycloid.

Then $\quad 4x = R\,(3 \cos \theta + \cos 3\theta) = 4a \cos^3 \theta$.

$\therefore\ x = a \cos^3 \theta$.

Similarly $y = a \sin^3 \theta$.

$\therefore\ x^{2/3} + y^{2/3} = a^{2/3}$.

This is known as the four cusped hypocycloid.

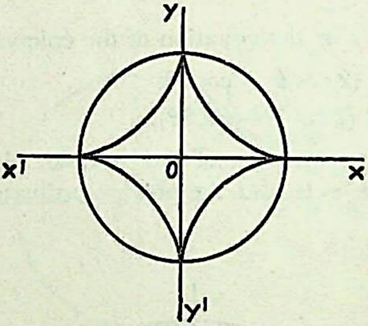

Fig. 53

(*c*) *Catenary*. The catenary is the curve in which a uniform heavy string will hang under the action of gravity.

Its cartesian equation is proved in books of Analytical Statics to be $y = c \cosh \frac{x}{c}$.

This form is that represented by the curve PCP'.

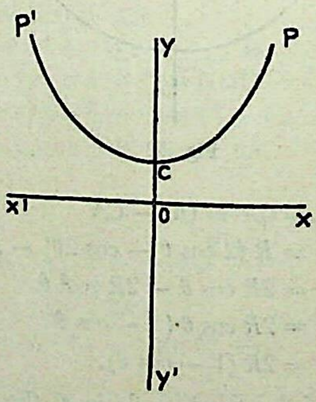

Fig. 54

INTEGRAL CALCULUS

CHAPTER XII

INTEGRATION

§ 42·1. We have so far considered the problem of differentiation, *viz.*, being given $y = f(x)$, find $\frac{dy}{dx}$. Now we pass on to the process, called integration, which may be regarded as the inverse of differentiation.

The problem is : Given $\frac{dy}{dx} = f(x)$, find y in terms of x.

This process of finding y is called *integration*. We write symbolically that $y = \int f(x)\, dx$. $\int$ is the sign of integration and the above statement is read as 'integral of $f(x)$ with respect to x' or shortly 'integral $f(x)\, dx$'. $f(x)$ is called the *integrand* and x is called the *variable of integration*. Here $\int f(x)\, dx$ is called the indefinite integral of $f(x)$ with respect to x or the *primitive** of $f(x)$ with respect to x and is to be distinguished from the definite integral to be explained in the succeeding pages. Hence by definition, the problem of evaluating $\int f(x)\, dx$ is to find $F(x)$, a function of x whose derivative with respect to x shall be the integrand $f(x)$, *i.e.*, $F'(x) = f(x)$.

An alternative method of defining an integral is to look upon it as the limit of the sum of a certain series. Many useful applications of calculus depend on this method but it is not convenient as the former in evaluating integrals. So we shall start with the first definition and consider its application to calculation of the various forms of integrals and then study the second definition in detail.

§ 42·2. Take for example $\int 2x\, dx$. We know that

$$\frac{d}{dx}(x^2) = 2x.$$

Hence by the first definition of an integral $\int 2x\, dx = x^2$. We may also add an arbitrary constant c to x^2 as

$$\frac{d}{dx}(x^2 + c) = 2x.$$

* Strictly speaking, we find only the 'primitive' of $f(x)$ but call it the 'integral' of $f(x)$ in a loose sense. There is a distinction between the two terms which we shall not go into here.

Hence $\int 2x\,dx = x^2 + c$. As the arbitrary constant of integration is present, this integral is called an *indefinite integral*.

Similarly $\int \sec x \tan x\,dx = \sec x + c$ as

$$\frac{d}{dx}(\sec x + c) = \sec x \tan x.$$

Thus in the above process, we depend on our knowledge of differentiation to guess at the function except for a numerical arbitrary constant which leads to the integrand by differentiation.

§ 43. The following list of formulae for integrals is based directly on the results of differentiation which have been studied earlier. It is necessary to commit them to memory.

1. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$ for all values of n except when $n = -1$.

2. In the case when $n = -1$,

$$\int \frac{dx}{x} = \log x + c.$$

(Hereafter, we shall take the constant c to be understood after the integral.)

3. $\int e^x\,dx = e^x$.
4. $\int \sin x\,dx = -\cos x$.
5. $\int \cos x\,dx = \sin x$.
6. $\int \sec^2 x\,dx = \tan x$.
7. $\int \operatorname{cosec}^2 x\,dx = -\cot x$.
8. $\int \sec x \tan x\,dx = \sec x$.
9. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x$.
10. $\int \cosh x\,dx = \sinh x$.
11. $\int \sinh x\,dx = \cosh x$.
12. $\int \dfrac{dx}{1+x^2} = \tan^{-1} x$.
13. $\int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$.
14. $\int \dfrac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x$.
15. $\int \dfrac{dx}{\sqrt{x^2-1}} = \sinh^{-1} x$.
16. $\int \dfrac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x$.

§ 44. Before we proceed to systematic methods of integration, we give below a few important results which can be easily proved from the definition of integration.

(1) $\int cf(x)\,dx = c\int f(x)\,dx$, where c is a constant.

(2) $\int (u \pm v)\,dx = \int u\,dx \pm \int v\,dx$, where u and v are functions of x.

Exercises XLVI.

Integrate the following with respect to x :—

1. x^{-4}.
2. $x^{3/2}$.
3. $ax + \dfrac{b}{x^2}$.
4. $\dfrac{ax^2 + bx + c}{x^3}$.
5. $\dfrac{ax^{-2} + bx^{-1} + c}{x^{-4}}$.
6. $\left(x + \dfrac{1}{x}\right)^2$.
7. $(x^{2/5} - x^{-3/5})^2$.
8. $x^2(1 - x)^2$.
9. $\dfrac{(x + 1)^4}{x^2}$.
10. $\dfrac{(1 - x^2)^2}{x}$.
11. $(x^2 - x^{-3/5})^2$.
12. $\dfrac{x + 3}{x\sqrt{x}}$.
13. $\dfrac{3x^2 + 4x - 5}{\sqrt{x}}$.
14. $\dfrac{(x^2 + 4x)(2x - 3)}{x^3}$.
15. $\tan^2 x$.
16. $\cot^2 x$.
17. $(\tan x - 2\cot x)^2$.
18. $\dfrac{1}{\sin^2 x \cos^2 x}$. $\left(Hint.\ \dfrac{1}{\sin^2 x \cos^2 x} = \dfrac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}.\right)$
19. $\dfrac{\sin^2 x}{1 + \cos x}$. (B.A. 50 M)
20. $\dfrac{\cos^2 x}{1 - \sin x}$.
21. $\dfrac{3}{\sqrt{1 - x^2}} + e^x + 8$.
22. $\sqrt{1 + \sin 2x}$.
23. $\dfrac{1}{1 + \sin x}$. $\left(Hint.\ \dfrac{1}{1 + \sin x} = \dfrac{1 - \sin x}{\cos^2 x} = \sec^2 x - \tan x \sec x.\right)$
24. $\dfrac{1}{1 - \sin x}$.
25. $\dfrac{1}{1 + \cos x}$.
26. $\dfrac{1}{1 - \cos x}$.

§ 45. Definite Integral.

Let $\int f(x)\,dx = F(x) + c$, where c is the arbitrary constant of integration. The value of the integral when $x = b$ is $F(b) + c$ and when $x = a$, the value is $F(a) + c$.

Subtracting

$$F(b) - F(a) = \text{the value of the integral when } x = b$$
$$- \text{the value of the integral when } x = a.$$

The symbol $\int_a^b f(x)\, dx$ denotes the value of the integral when $x = b$, minus the value of the integral when $x = a$ and is thus $F(b) - F(a)$. $\int_a^b f(x)\, dx$ is called the definite integral; a and b are called the limits of integration, a being the lower limit and b the upper limit.

Note. $\int_a^b f(x)\, dx$ is a definite constant unlike $\int_a^x f(x)\, dx$ which is a function of the variable x. $\int_a^x f(x)\, dx$ is called an indefinite integral in quite a different sense. The upper limit here is x, a variable and not a constant. For this reason this integral is called an indefinite integral.

Rule to find $\int_a^b f(x)\, dx$.

Evaluate the indefinite integral of f (x) with respect to x. Let it be F (x). Subtract the value of F (x) when x = a from its value when x = b. The result obtained is $\int_a^b f(x)\, dx$.

Examples.

Ex. 1. $\int^2 \left(x^2 - 3x^{1/2} + \frac{1}{x^2}\right) dx$

$$= \left[\frac{x^3}{3} - 2x^{3/2} - \frac{1}{x}\right]_1^2$$

$$= \left(\frac{8}{3} - 4\sqrt{2} - \frac{1}{2}\right) - \left(\frac{1}{3} - 2 - 1\right)$$

$$= \frac{29}{6} - 4\sqrt{2}.$$

Ex. 2. $\int_0^{\pi/6} \cos^2 \frac{x}{2}\, dx = \frac{1}{2}\int_0^{\pi/6} (1 + \cos x)\, dx$

$$= \frac{1}{2}\Big[x + \sin x\Big]_0^{\pi/6}$$

$$= \frac{1}{2}\Big[\left(\frac{\pi}{6} + \sin\frac{\pi}{6}\right) - 0\Big]$$

$$= \frac{\pi}{12} + \frac{1}{4}.$$

§ 46. Methods of Integration.

The various rules in the differential calculus enable us to differentiate almost any combination of the various ordinary functions. But it is not so with integration. In fact the integrals of some even fairly simple functions cannot be found in terms of the functions which are known to the students at this stage. For example,

$$(a + b \sin^2 x)^{1/2}, \quad \sqrt{1 - x^3}, \quad \sqrt{\sin x}, \quad \frac{\cos x}{x}$$

cannot be integrated in terms of functions which are known.

Corresponding to the various rules in the differential calculus for differentiating sums, products and functions of functions, we have more or less similar rules in the integral calculus. These give rise to the following methods of integration :—

1. Substitution.
2. Decomposition into a sum.
3. Integration by parts.
4. Successive reduction.

§ 47. The efficacy of the method of substitution depends on finding a suitable substitution to convert the given integral into a standard form. The form of the integrand often suggests the proper substitution.

To evaluate $\int f(x)\, dx$, we put $x = \phi(t)$.

$$\frac{dx}{dt} = \phi'(t) \text{ or } dx = \phi'(t)\, dt.$$

Then $\int f(x)\, dx = \int f\{\phi(t)\}\, \phi'(t)\, dt.$

To prove this, $\frac{d}{dx}$ (left-hand side) $= f(x)$ by definition and

$$\frac{d}{dx}\text{(right-hand side)} = \frac{d}{dt}\text{(right-hand side)} \times \frac{dt}{dx}$$
$$= f\{\phi(t)\}\phi'(t) \times \frac{1}{\phi'(t)}$$
$$= f(x).$$

Hence the result.

§ 47·1. Integrals of functions containing linear functions of x, *i.e.*, f (ax + b).

Put $ax + b = t$. $\therefore$ $a\,dx = dt$.

$$\therefore \int f(ax+b)\,dx = \int f(t)\,.\frac{1}{a}\,dt$$
$$= \frac{1}{a}\int f(t)\,dt \text{ which can be evaluated.}$$

Examples.

Ex. 1. $\int (ax+b)^n\,dx \; (n \neq -1)$.

Put $t = ax + b$, then $dt = a\,dx$.

$$\therefore \int (ax+b)^n\,dx = \frac{1}{a}\int t^n\,dt = \frac{t^{n+1}}{a(n+1)}$$
$$= \frac{(ax+b)^{n+1}}{a(n+1)}.$$

Similarly

2. $\displaystyle\int \frac{dx}{ax+b} = \frac{1}{a}\log(ax+b)$.

3. $\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b}$.

4. $\displaystyle\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)$.

5. $\displaystyle\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)$.

6. $\displaystyle\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)$.

7. $\displaystyle\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b)$.

8. $\displaystyle\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b)$.

9. $\displaystyle\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b)$

Ex. 2. Evaluate $\int \frac{x^2}{(a+bx)^3}\,dx.$

Put $a+bx=t.\quad \therefore\quad b\,dx=dt.$

$$\therefore \int \frac{x^2}{(a+bx)^3}\,dx=\int \frac{\left(\frac{t-a}{b}\right)^2\frac{1}{b}\,dt}{t^3}$$

$$=\frac{1}{b^3}\int \frac{(t-a)^2}{t^3}\,dt$$

$$=\frac{1}{b^3}\int \left(\frac{1}{t}-\frac{2a}{t^2}+\frac{a^2}{t^3}\right)dt$$

$$=\frac{1}{b^3}\log t+\frac{2a}{b^3}\cdot\frac{1}{t}-\frac{a^2}{2b^3}\cdot\frac{1}{t^2}$$

$$=\frac{1}{b^3}\log(a+bx)+\frac{2a}{b^3}\cdot\frac{1}{a+bx}-\frac{a^2}{2b^3}\cdot\frac{1}{(a+bx)^2}.$$

Ex. 3. Evaluate $\int \cos mx\cos nx\,dx.$

Case (*i*). $m\neq n.$

$$\int \cos mx\cos nx\,dx=\tfrac{1}{2}\int\{\cos(m+n)x+\cos(m-n)x\}\,dx$$
$$=\frac{1}{2}\left\{\frac{\sin(m+n)x}{m+n}+\frac{\sin(m-n)x}{m-n}\right\}.$$

Case ii. $m=n.$

The integral is $\int \cos^2 mx\,dx=\tfrac{1}{2}\int(1+\cos 2mx)\,dx$

$$=\frac{1}{2}\left(x+\frac{\sin 2mx}{2m}\right).$$

Similarly

$$\int \sin mx\sin nx\,dx=-\frac{\sin(m+n)x}{2(m+n)}+\frac{\sin(m-n)x}{2(m-n)}$$
if $m\neq n.$

$$\int \sin^2 mx\,dx=\frac{1}{2}\left(x-\frac{\sin 2mx}{2m}\right).$$

$$\int \sin mx\cos nx\,dx=-\frac{\cos(m+n)x}{2(m+n)}-\frac{\cos(m-n)x}{2(m-n)}$$
if $m\neq n.$

Ex. 4. Evaluate $\int \sin^2 3x\,dx.$

$$\int \sin^2 3x\,dx=\frac{1}{2}\int(1-\cos 6x)\,dx=\frac{1}{2}\left(x-\frac{\sin 6x}{6}\right).$$

Ex. 5. Evaluate $\int \cos^3 x\, dx$.

$$\int \cos^3 x\, dx = \int \frac{\cos 3x + 3\cos x}{4}\, dx$$

$$= \frac{1}{12}\sin 3x + \frac{3}{4}\sin x.$$

Ex. 6. Evaluate $\int \sin^4 x\, dx$.

$$\int \sin^4 x\, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \int \left(\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right) dx$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x.$$

Exercises XLVII.

Integrate the following expressions :—

1. x^3, $(2+x)^3$, $(3-2x)^3$, $(3x-4)^3$, $(ax+b)^3$, $(b-a$
2. x^n, $(x+a)^n$, $(3x+2)^n$, $(3-2x)^n$, $(ax-b)^n$, $(b-a$
3. $\sqrt{x}$, $\sqrt{1+x}$, $\sqrt{2+3x}$, $\sqrt{4-5x}$, $\sqrt{a+}$ $\sqrt{a+\frac{bx}{c}}$.
4. $\frac{1}{x^4}$, $\frac{1}{(x+7)^4}$, $\frac{1}{(2x+3)^4}$, $\frac{1}{(4-3x)^4}$, $\frac{1}{(a+b}$ $\frac{1}{(a-bx)^4}$.
5. $\frac{1}{x^{3/2}}$, $\frac{1}{(2x+1)^{3/2}}$, $\frac{1}{(3x-4)^{3/2}}$, $\frac{1}{(2-3x)^{3/2}}$, $\frac{1}{(a+bx}$ $\frac{1}{(a-bx)^{3/2}}$.
6. $\frac{1}{2x}$, $\frac{1}{ax}$, $\frac{1}{3x+7}$, $\frac{1}{2-7x}$, $\frac{1}{ax+b}$, $\frac{1}{b-ax}$.
7. e^{4x}, e^{3x+7}, e^{2-3x}, $e^{(x-4)/3}$, e^{ax+b}.
8. $\sin 2x$, $\sin\frac{x}{2}$, $\sin(2x+3)$, $\sin(3-2x)$.
9. $\cos 3x$, $\cos\frac{x}{3}$, $\cos(3x+2)$, $\cos(2-3x)$.
10. $\sec^2 4x$, $\sec^2\frac{x}{4}$, $\sec^2(4x-7)$, $\sec^2(7-4x)$.
11. $\operatorname{cosec}^2 5x$, $\operatorname{cosec}^2\frac{x}{5}$, $\operatorname{cosec}^2(5x+3)$, $\operatorname{cosec}^2(3-5x)$.

12. $\tan 2x \sec 2x$, $\tan \frac{x}{2} \sec \frac{x}{2}$, $\tan (2x-3) \sec (2x-3)$.

13. $\cot 3x \operatorname{cosec} 3x$, $\cot \frac{x}{3} \operatorname{cosec} \frac{x}{3}$, $\cot (b-ax) \operatorname{cosec} (b-ax)$.

14. $x\sqrt{x+a}$.

15. $\dfrac{x}{(a+bx)^{1/2}}$.

16. $\dfrac{x}{(a+bx)^{1/3}}$.

17. $\dfrac{1}{\sqrt{x+a}+\sqrt{x}}$.

18. $\dfrac{x}{(a+bx)^n}$.

19. $\dfrac{a+bx}{a_1+b_1x}$.

20. $\dfrac{1-\cos x}{1+\cos x}$.

21. $\dfrac{\sin^2 x}{(1+\cos x)^2}$.

22. $\dfrac{\cos x}{1+\cos x}$.

23. $\sin 3x \cos 2x$.

24. $\sin 3x \sin 6x$.

25. $\cos 4x \cos 5x$.

26. $\cos^2 4x$.

27. $\sin^2 (2x+5)$.

28. $\sin x \cdot \sin 2x \cdot \sin 3x$.

29. $\cos 2x \cdot \cos 4x \cdot \cos 6x$.

30. $\sin^2 x \cos^2 x$.

31. $\sin^3 x$.

32. $\cos^4 x$.

33. $\sin^2 x \cos 3x$. (B.A. Sub. 38)

34. $\sin^3 x \cos^3 x$. (B.Sc. 45 M)

35. $\tan^2 3x$.

36. Evaluate $\int_0^{\pi/4} \sin^4 x\, dx$.

§ 47·2. Integrals of functions involving $a^2 \pm x^2$.

To evaluate $\int \dfrac{dx}{\sqrt{a^2-x^2}}$, put $x = a\sin\theta$.

Then $dx = a\cos\theta\, d\theta$.

$$\therefore \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{a\cos\theta} = \int d\theta = \theta$$

$$= \left(\sin^{-1}\frac{x}{a}\right)$$

Similarly $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$.

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right).$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right).$$

Again
$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\int\left(\frac{1}{a+x} + \frac{1}{a-x}\right)dx$$
$$= \frac{1}{2a}\{\log(a+x) - \log(a-x)\}$$
$$= \frac{1}{2a}\log\frac{a+x}{a-x}.$$

Similarly
$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}.$$

Examples.

1. $\displaystyle\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int\frac{dx}{\sqrt{\frac{4}{9}-x^2}} = \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right).$

2. $\displaystyle\int \frac{dx}{\sqrt{9+4x^2}} = \frac{1}{2}\int\frac{dx}{\sqrt{\frac{9}{4}+x^2}} = \frac{1}{2}\sinh^{-1}\left(\frac{2x}{3}\right).$

3. $\displaystyle\int \frac{dx}{4+9x^2} = \frac{1}{9}\int\frac{dx}{\frac{4}{9}+x^2} = \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right).$

4. $\displaystyle\int \frac{dx}{9x^2-4} = \frac{1}{9}\int\frac{dx}{x^2-\frac{4}{9}}$

$$= \frac{1}{9}\cdot\frac{1}{2\left(\frac{2}{3}\right)}\log\frac{x-\frac{2}{3}}{x+\frac{2}{3}} = \frac{1}{12}\log\frac{3x-2}{3x+2}.$$

5. $\displaystyle\int \frac{dx}{4-25x^2} = \frac{1}{25}\int\frac{dx}{\frac{4}{25}-x^2}$

$$= \frac{1}{25}\cdot\frac{1}{2\left(\frac{2}{5}\right)}\log\frac{\frac{2}{5}+x}{\frac{2}{5}-x}$$
$$= \frac{1}{20}\log\frac{2+5x}{2-5x}.$$

6. $\displaystyle\int \frac{dx}{\sqrt{4x^2-9}} = \frac{1}{2}\int\frac{dx}{\sqrt{x^2-\frac{9}{4}}} = \frac{1}{2}\cosh^{-1}\left(\frac{2x}{3}\right).$

Exercises XLVIII.

Integrate the following expressions :—

1. $\dfrac{1}{1+4x^2},\quad \dfrac{1}{1+\dfrac{x^2}{4}},\quad \dfrac{1}{1+7x^2},\quad \dfrac{1}{1+2(x+2)^2},\quad \dfrac{1}{a^2+b^2x^2}.$

2. $\dfrac{1}{(x+2)^2+3},\quad \dfrac{1}{(2-x)^2+16},\quad \dfrac{1}{(3x+2)^2+16},\quad \dfrac{1}{(2x+3)^2+9}.$

3. $\dfrac{1}{1-4x^2},\quad \dfrac{1}{1-\dfrac{x^2}{4}},\quad \dfrac{1}{1-7x^2},\quad \dfrac{1}{1-2(x+2)^2},\quad \dfrac{1}{a^2-b^2x^2}.$

4. $\dfrac{1}{9x^2-4}$, $\dfrac{1}{x^2-36}$, $\dfrac{1}{4x^2-9}$, $\dfrac{1}{bx^2-a}$, $\dfrac{1}{ax^2-b^2}$.

5. $\dfrac{1}{\sqrt{4-x^2}}$, $\dfrac{1}{\sqrt{1-4x^2}}$, $\dfrac{1}{\sqrt{9-25x^2}}$, $\dfrac{1}{\sqrt{a^2-b^2x^2}}$, $\dfrac{1}{\sqrt{a-bx^2}}$.

6. $\dfrac{1}{\sqrt{x^2-1}}$, $\dfrac{1}{\sqrt{4x^2-1}}$, $\dfrac{1}{\sqrt{b^2x^2-a^2}}$, $\dfrac{1}{\sqrt{(2-x)^2-25}}$.

7. $\dfrac{1}{\sqrt{4+x^2}}$, $\dfrac{1}{\sqrt{1+4x^2}}$, $\dfrac{1}{\sqrt{9+25x^2}}$, $\dfrac{1}{\sqrt{a^2+b^2x^2}}$,

$\dfrac{1}{\sqrt{7+(3-2x)^2}}$.

§ 47·3. Integrals of functions of the form $\int f(x^n) \,.\, x^{n-1}\,dx$.

Put $x^n = t$, then the integral reduces to $\frac{1}{n}\int f(t)\,dt$ which can be evaluated.

Examples.

Ex. 1. $\int x^2 \cos(x^3)\,dx$. Put $x^3 = t$, $3x^2\,dx = dt$.

$$\therefore \int x^2 \cos(x^3)\,dx = \frac{1}{3}\int \cos t\,dt = \frac{\sin t}{3} = \frac{\sin(x^3)}{3}.$$

Ex. 2. $\displaystyle\int \frac{x^3}{\sqrt{1-x^8}}\,dx$. Put $x^4 = t$; $4x^3dx = dt$.

$$\therefore \int \frac{x^3}{\sqrt{1-x^8}}\,dx = \frac{1}{4}\int \frac{dt}{\sqrt{1-t^2}} = \tfrac{1}{4}\sin^{-1}(t)$$
$$= \tfrac{1}{4}\sin^{-1}(x^4).$$

Ex. 3. $\displaystyle\int \frac{3x}{1+2x^4}\,dx$. Put $x^2 = t$, $2x\,dx = dt$.

$$\therefore \int \frac{3x}{1+2x^4}\,dx = \frac{3}{2}\int \frac{dt}{1+2t^2}$$
$$= \frac{3}{4}\int \frac{dt}{\frac{1}{2}+t^2}$$
$$= \tfrac{3}{4}\,.\,\sqrt{2}\tan^{-1}(\sqrt{2}\,t)$$
$$= \frac{3\sqrt{2}}{4}\tan^{-1}(\sqrt{2}\,x^2).$$

Exercises XLIX.

Integrate the following expressions :—

1. $x^{n-1}\sin(x^n)$. 2. $\dfrac{\sin\sqrt{x}}{\sqrt{x}}$. 3. $\dfrac{x^2}{a^6+x^6}$.

4. $\dfrac{x^2}{1-x^6}$. 5. $\dfrac{x^2}{\sqrt{x^6+1}}$. 6. $\dfrac{x}{x^4+a^4}$.

7. $\dfrac{x}{\sqrt{1-x^4}}$. 8. $\dfrac{x^3}{\sqrt{a^8-x^8}}$. 9. $\dfrac{x^2}{\sqrt{a^3-x^3}}$.

10. $\dfrac{x^2}{1+x^6}$. 11. $\dfrac{x^2}{\sqrt{1-x^6}}$. 12. $\dfrac{x^3}{x^2+a^2}$.

13. $\dfrac{x^3+x}{x^4-9}$. 14. $\dfrac{x+1}{\sqrt{x^2+1}}$.

15. $\dfrac{4x^3+3x}{x^4+1}$. (B.Sc. Sub. 47)

16. Evaluate $\displaystyle\int_0^2 \frac{5x+1}{x^2+4}\,dx$. (B.A. Sub. 45)

17. $\displaystyle\int_0^\infty xe^{-x^2}\,dx$. (B.Sc. Sub. 39)

§ 47·4. Integrals of functions of the form

$$\int \{f(x)\}^n f'(x)\,dx.$$

When $n \neq -1$, put $f(x) = t$, then $f'(x)\,dx = dt$.

$$\therefore \int \{f(x)\}^n f'(x)\,dx = \int t^n\,dt = \frac{t^{n+1}}{n+1}$$

$$= \frac{\{f(x)\}^{n+1}}{n+1}.$$

When $n = -1$, the integral reduces to

$$\int \frac{f'(x)}{f(x)}\,dx.$$

Putting $y = f(x)$, the above integral reduces to

$$\int \frac{dy}{y} = \log y = \log f(x).$$

Examples.

Ex. 1. $\int \sqrt{x^2+a^2}\,x\,dx$.

Here the derivative of x^2+a^2 is $2x$.

Hence put $x^2+a^2 = t$. $\therefore 2x\,dx = dt$.

$$\therefore \int \sqrt{x^2+a^2}\,x\,dx = \frac{1}{2}\int \sqrt{t}\,dt = \frac{1}{3}t^{3/2}$$

$$= \frac{1}{3}(x^2+a^2)^{3/2}.$$

Ex. 2. $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx.$ Putting $t = \sin^{-1} x.$

$$dt = \frac{1}{\sqrt{1-x^2}} dx.$$

$\therefore$ The integral reduces to $\int t\,dt = \frac{t^2}{2} = \frac{1}{2}(\sin^{-1} x)^2.$

Ex. 3. $\int \tan\theta\, d\theta = \int \frac{\sin\theta}{\cos\theta} d\theta$

$$= -\int \frac{dy}{y} \text{ on putting } y = \cos\theta$$

$$= -\log y$$

$$= -\log\cos\theta = \log(\sec\theta).$$

Ex. 4. $\int \cot\theta\, d\theta = \int \frac{\cos\theta}{\sin\theta} d\theta$

$$= \int \frac{dy}{y} \text{ on putting } y = \sin\theta$$

$$= \log y = \log\sin\theta.$$

Ex. 5. $\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$

$$= \int \frac{dy}{y} \text{ where } y = \sec x + \tan x$$

$$= \log y = \log(\sec x + \tan x)$$

$$= \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

Ex. 6. $\int \operatorname{cosec} x\, dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x + \cot x)\, dx}{(\operatorname{cosec} x + \cot x)}$

$$= -\int \frac{d(\operatorname{cosec} x + \cot x)}{(\operatorname{cosec} x + \cot x)}$$

$$= -\log(\operatorname{cosec} x + \cot x)$$

$$= \log\tan x/2.$$

Exercises L.

Integrate the following expressions :—

1. $(x^2+1)^{7/2} x.$
2. $\frac{x}{(1+x^2)^{3/2}}.$
3. $5x\sqrt{1-2x^2}.$
4. $\frac{x}{(1-x^2)^2}.$
5. $x(a^2+x^2)^{3/2}.$
6. $x^2\sqrt{a^3+x^3}.$
7. $x(a-bx^2)^n.$
8. $\frac{x^5}{a^6+x^6}.$
9. $(e^x+k)^n e^x.$

10. $\dfrac{x^2}{1-x^3}$.

11. $\dfrac{1}{x}\log x$.

12. $\dfrac{1}{x\log x}$.

13. $\dfrac{1}{x(\log x)^n}$.

14. $\dfrac{(1+\log x)^n}{x}$.

15. $\dfrac{(1+\sqrt{x})^n}{\sqrt{x}}$.

16. $\dfrac{1+\cos x}{(x+\sin x)^2}$.

17. $\dfrac{3x+2}{(3x^2+4x+2)^4}$.

18. $\tan^n x \,.\, \sec^2 x$.

19. $\sec^2 x\,(1-\tan x)^n$.

20. $\cot^n x \operatorname{cosec}^2 x$.

21. $\dfrac{\sin x}{\cos^n x}$.

22. $\sin x \,.\, \sec^4 x$.

23. $\dfrac{\sin x}{(1+\cos x)^2}$.

24. $\dfrac{e^x}{e^x+1}$.

25. $\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$.

26. $\dfrac{1}{(1+x^2)\tan^{-1}x}$.

27. $\dfrac{1}{\sqrt{1-x^2}}\dfrac{1}{\sin^{-1}x}$.

28. $\dfrac{1+\cos x}{x+\sin x}$.

29. $\dfrac{e^{2x}}{e^{2x}-1}$. (B.Sc. Sub. 44)

30. $\dfrac{x^{n-1}}{a+bx^n}$.

31. $\dfrac{\cos x}{a+b\sin x}$.

32. $\dfrac{\sin x}{a+b\cos x}$.

33. $\dfrac{\cot x}{\log\sin x}$.

34. $\dfrac{\tan x}{\log\cos x}$.

35. $\dfrac{\sec^2 x}{5+4\tan x}$.

36. $\dfrac{x^2-2}{x^3-6x+4}$.

37. $\left(\dfrac{1}{a}-\dfrac{1}{x}\right)^n\dfrac{1}{x^2}$.

38. $\dfrac{e^x}{(2e^x-3)^2}$.

39. $\cot x+\sin x$. (B.A. 39 M

40. $\sec 4x$.

41. $\dfrac{\cos 2x}{\cos x}$.

42. $\displaystyle\int_0^1 \frac{x\,dx}{1+\sqrt{x}}$. (B.Sc. Anc. 59

43. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\sin x}\,dx$. (B.Sc. Anc. 5

44. $\displaystyle\int \frac{\tan^m x}{\sin x\cos x}\,dx \;(m\neq 0)$. (B.Sc. Comp. 6

(*Hint* : Put $t=\tan x$.)

§ 47·5. Integrals of functions of the form

$$\int F\{f(x)\}f'(x)\,dx.$$

Putting $f(x) = y$, the above integral reduces to $\int F(y)\,dy$.

Examples.

Ex. 1. $\int x^2\sqrt{1-4x^3}\,dx$. Putting $1-4x^3 = y$

$$-12x^2\,dx = dy.$$

The integral $= -\int \frac{\sqrt{y}}{12}\,dy = -\frac{y^{3/2}}{18} = -\frac{(1-4x^3)^{3/2}}{18}$.

Ex. 2. $\int \frac{e^x}{e^{x/2}-1}\,dx$. (B.A. 46 M)

$$= \int \frac{e^{x/2}\,.\,e^{x/2}\,dx}{e^{x/2}-1}. \quad \text{Put } y = e^{x/2}-1\,;\ dy = \tfrac{1}{2}e^{x/2}\,dx$$

$$= \int \frac{(y+1)\,2\,dy}{y}$$

$$= 2\int\left(1+\frac{1}{y}\right)dy$$

$$= 2\,(y + \log y)$$

$$= 2\,\{e^{x/2} - 1 + \log\,(e^{x/2}-1)\}.$$

Ex. 3. $\int \frac{dx}{(1+e^x)(1+e^{-x})}$. (B.A. 49 M)

$$= \int \frac{e^x\,dx}{(1+e^x)^2}$$

$$= \int \frac{dy}{y^2} \text{ where } y = 1+e^x$$

$$= -\frac{1}{y} = -\frac{1}{1+e^x}.$$

Ex. 4. $\int \frac{dx}{\sin x\cos^2 x} = \int \frac{\sin^2 x+\cos^2 x}{\sin x\cos^2 x}\,dx.$

$$= \int \frac{\sin x}{\cos^2 x}\,dx + \int \frac{dx}{\sin x}$$

$$= \int \tan x\sec x\,dx + \int \frac{dx}{\sin x}$$

$$= \sec x + \log\tan x/2.$$

Ex. 5. $\int \frac{\tan x\,dx}{\sec x+\cos x}$. (B.A. 50 M)

$$= \int \frac{\sin x\,dx}{1+\cos^2 x}$$

$$= -\int \frac{dt}{1+t^2} \text{ where } t = \cos x$$

$$= -\tan^{-1}(t) = -\tan^{-1}(\cos x).$$

Ex. 6. $\int \frac{2\cos x + 3\sin x}{4\cos x + 5\sin x}\,dx.$ (B.A. 49 M)

$\frac{d}{dx}$ (denominator) $= -4\sin x + 5\cos x.$

Putting the numerator $2\cos x + 3\sin x$

$$= l(4\cos x + 5\sin x) + m(-4\sin x + 5\cos x)$$

and equating the coefficients of $\sin x$ and $\cos x$, we have $4l + 5m = 2$ and $5l - 4m = 3$.

$$\therefore \quad l = \frac{23}{41} \text{ and } m = -\frac{2}{41}.$$

Hence the integral reduces to

$$\frac{23}{41}\int dx - \frac{2}{41}\int \frac{d(4\cos x + 5\sin x)}{4\cos x + 5\sin x}\,dx$$

$$= \frac{23}{41}x - \frac{2}{41}\log(4\cos x + 5\sin x).$$

Ex. 7. $\int \frac{dx}{1+\tan x}.$ (B.Sc. 49 M)

$$= \int \frac{\cos x}{\sin x + \cos x}\,dx$$

$$= \frac{1}{2}\int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{\sin x + \cos x}\,dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{d(\cos x + \sin x)}{\sin x + \cos x}\,dx$$

$$= \tfrac{1}{2}x + \tfrac{1}{2}\log(\sin x + \cos x).$$

Exercises LI.

Integrate the following expressions :—

1. $\frac{\sin(\log x)}{x}$.

4. $\frac{\sin(\tan^{-1} x)}{1+x^2}$.

2. $\frac{\sec^2(\log x)}{x}$.

5. $\frac{\sin x}{1+9\cos^2 x}$.

3. $\frac{e^{\tan^{-1} x}}{1+x^2}$.

6. $\frac{\cos x}{4+\sin^2 x}$.

7. $e^{\sin^2 x + \cos x}(\sin 2x - \sin x)$. (B.Sc. Sub.

8. $\frac{\sin x}{\sqrt{2-\cos^2 x}}$.

9. $\frac{x+2}{\sqrt{1-x^2}}$.

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10. $\dfrac{1}{\sqrt{1-x^2}} \dfrac{1}{(\sin^{-1} x)^2}$.

11. $\sin^4 x \cos^3 x$.

12. $\dfrac{\cos^3 x}{\sqrt{\sin x}}$.

13. $\dfrac{\sin^3 x}{\cos^4 x}$.

14. $\dfrac{\cos^5 x}{\sin^2 x}$.

15. $(\sin x)^{5/2} \cos^3 x$.

(B.Sc. Sub. 41)

16. $\dfrac{\sin^8 x}{\cos^{10} x}$.

17. $\dfrac{\cos^9 x}{\sin x}$.

18. $\sqrt{\cos \theta} \sin^3 \theta$.

19. $\sin^5 x$.

20. $\cos^5 x$.

21. $\sin^7 x$.

22. $\sec^4 x$.

23. $\tan^4 x$.

24. $\cot^4 x$.

25. $\tan^3 x$.

26. $\tan^5 x$.

27. $\cot^5 x$.

28. $\sec x \operatorname{cosec} x$.

29. $\sec^2 x \operatorname{cosec}^2 x$.

30. $\dfrac{\sin x \cos x}{a^2 \cos^2 x + b^2 \sin^2 x}$.

31. $\dfrac{\sin 2x}{2 + 3 \cos^2 x}$.

32. $\dfrac{1}{e^{-x} + 1}$.

33. $\dfrac{1}{e^x + 1}$.

34. $\dfrac{2 \sin x + \cos x}{3 \sin x + \cos x}$.

(B.A. 43 M)

35. $\dfrac{5 \cos \theta}{2 \cos \theta + \sin \theta}$.

(B.A. 55 M)

36. $\dfrac{a}{b + ce^x}$.

37. $\dfrac{\sec x}{(\sec x + \tan x)^n}$ $(n > 0)$.

38. $\dfrac{\sec^2 x}{(\sec x + \tan x)^n}$ $(n > 1)$.

39. Show that $\displaystyle\int \frac{\cos^2 x}{\sin^6 x} dx = -\frac{\cot^3 x}{3} - \frac{\cot^5 x}{5}$.

40. $\displaystyle\int \sec^{3/5} x \operatorname{cosec}^{7/5} x \, dx = -\frac{5}{2} \cot^{2/5} x$.

41. $\displaystyle\int \frac{dx}{\sin x \cos^5 x} = \log \tan x + \tan^2 x + \frac{\tan^4 x}{4}$.

42. $\displaystyle\int_0^{\pi/2} \sqrt{\sin \theta} \, . \cos^5 \theta \, d\theta = \frac{64}{231}$. (B.A. 50 M)

43. $\displaystyle\int_0^{\pi/2} \frac{\cos \theta}{1 + \cos^2 \theta} d\theta = \frac{1}{\sqrt{2}} \log (\sqrt{2} + 1)$.

44. $\displaystyle\int_0^{\pi} \sin^3 x \cos^4 x \, dx = \frac{4}{35}$.

45. Evaluate $\int_0^{\pi/4} \sin^5 \theta \cos^2 \theta \, d\theta$. (B.Sc. 50 M)

46. Evaluate $\int_0^{\pi/2} \sin^3 x \cos^2 x \, dx$. (B.Sc. 47 M)

47. Evaluate $\int_{\pi/4}^{\pi/2} \cos^3 \theta \operatorname{cosec} \theta \, d\theta$. (B.A. 45 S)

48. Evaluate $\int_0^1 \frac{dx}{e^x + e^{-x}}$.

49. Evaluate $\int_0^\infty \frac{dx}{a^2 e^x + b^2 e^{-x}}$.

50. Evaluate $\int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$.

51. $\int_0^{\pi/2} \sin^{11/2} x \cos^5 x \, dx$. (B.Sc. 60)

§ 48. Integration of rational algebraic functions.

We proceed to integrate fractions whose numerator and denominator contain positive integral powers of x with constant coefficients.

§ 48·1. **Rule (a).** If the degree of the numerator is equal to or greater than the degree of the denominator, divide the numerator by the denominator until the remainder is of the lower degree than the denominator.

Examples.

Ex. 1. $\int \frac{x^2}{x+2} dx$.

$$\frac{x^2}{x+2} = x - 2 + \frac{4}{x+2} \text{ by division.}$$

$$\therefore \int \frac{x^2}{x+2} dx = \int \left(x - 2 + \frac{4}{x+2} \right) dx$$

$$= \frac{x^2}{2} - 2x + 4 \log (x+2).$$

Ex. 2. $$\int \frac{2+3x}{3-4x}\,dx = \int \left(-\frac{3}{4} + \frac{17}{4}\cdot\frac{1}{3-4x}\right) dx$$
$$= -\frac{3x}{4} - \frac{17}{16}\log(3-4x).$$

Ex. 3. $$\int \frac{x^{24}}{x^{10}+1}\,dx.$$ (B.A. 46 M)

$$\int \frac{x^{24}}{x^{10}+1}\,dx = \int \left(x^{14} - x^4 + \frac{x^4}{x^{10}+1}\right) dx$$
$$= \frac{x^{15}}{15} - \frac{x^5}{5} + \int \frac{x^4\,dx}{x^{10}+1}$$
$$= \frac{x^{15}}{15} - \frac{x^5}{5} + \frac{1}{5}\int \frac{dy}{y^2+1},\ \text{putting } x^5 = y$$
$$= \frac{x^{15}}{15} - \frac{x^5}{5} + \frac{1}{5}\tan^{-1}(y)$$
$$= \frac{x^{15}}{15} - \frac{x^5}{5} + \frac{1}{5}\tan^{-1}(x^5).$$

Exercises LII.

Integrate

1. $\dfrac{x^2-x+3}{x-1}$.
2. $\dfrac{x}{3+2x}$.
3. $\dfrac{lx+m}{ax+b}$.
4. $\dfrac{x^2}{2x+1}$.
5. $\dfrac{x^3}{1-x}$.
6. $\dfrac{x^4}{x^2+3}$.
7. $\dfrac{x^2+2x+5}{x^2+1}$.
8. $\dfrac{x^3+1}{x^2+1}$.
9. $\dfrac{x^8}{x^6+1}$.
10. $\dfrac{x^{27}}{x^{14}+4}$.
11. $\dfrac{x^4+x^2+1}{x^2-1}$.
12. $\dfrac{\sin^2 x\cos x}{1-\sin x}$. (B.A. 54 M)
13. $\dfrac{\sin x\cos x}{1-a\cos x}$. (B.A. 54 S)
14. Show that
$$\int_0^1 \frac{x^4\,dx}{1+x^2} = \frac{\pi}{4} - \frac{2}{3}.$$

48·2. Rule (b). Denominator is of the second degree and does not resolve into rational factors. It has been shown that

(1) $$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right).$$

(2) $$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}.$$

$$(3)\quad \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a + x}{a - x}.$$

Type i : $\displaystyle\int \frac{dx}{ax^2 + bx + c}.$

Then dividing the denominator by the coefficient of x^2 and completing the square of the term which contains x, the integral reduces to one of the three forms just mentioned.

Examples.

Ex. 1. $$\int \frac{dx}{x^2 + 2x + 5} = \int \frac{dx}{(x+1)^2 + 2^2} = \frac{1}{2} \tan^{-1}\left(\frac{x+1}{2}\right)$$

Ex. 2. $$\int \frac{dx}{4x^2 - 4x + 2} = \frac{1}{4} \int \frac{dx}{x^2 - x + \frac{1}{2}}$$
$$= \frac{1}{4} \int \frac{dx}{(x - \frac{1}{2})^2 + \frac{1}{4}}$$
$$= \frac{1}{4} \cdot \frac{1}{2} \tan^{-1}\left(\frac{x - \frac{1}{2}}{\frac{1}{2}}\right)$$
$$= \frac{1}{8} \tan^{-1}(2x - 1).$$

Ex. 3. $$\int \frac{dx}{x^2 + 8x - 7} = \int \frac{dx}{(x+4)^2 - 23}$$
$$= \frac{1}{2\sqrt{23}} \log \frac{x + 4 - \sqrt{23}}{x + 4 + \sqrt{23}}.$$

Ex. 4. $$\int \frac{dx}{3x^2 - 4x - 5} = \frac{1}{3} \int \frac{dx}{x^2 - \frac{4}{3}x - \frac{5}{3}}$$
$$= \frac{1}{3} \int \frac{dx}{(x - \frac{2}{3})^2 - \frac{19}{9}}$$
$$= \frac{1}{\frac{3 \cdot 2\sqrt{19}}{3}} \log \frac{x - \frac{2}{3} - \frac{\sqrt{19}}{3}}{x - \frac{2}{3} + \frac{\sqrt{19}}{3}}$$
$$= \frac{1}{2\sqrt{19}} \log \frac{3x - 2 - \sqrt{19}}{3x - 2 + \sqrt{19}}.$$

Ex. 5. $$\int \frac{dx}{1 + x - x^2} = \int \frac{dx}{\frac{5}{4} - (x - \frac{1}{2})^2}$$
$$= \frac{1}{\frac{2\sqrt{5}}{2}} \log \frac{\frac{\sqrt{5}}{2} + x - \frac{1}{2}}{\frac{\sqrt{5}}{2} - \left(x - \frac{1}{2}\right)}$$
$$= \frac{1}{\sqrt{5}} \log \frac{\sqrt{5} - 1 + 2x}{\sqrt{5} + 1 - 2x}.$$

Ex. 6. $$\int \frac{dx}{1-6x-9x^2} = \frac{1}{9}\int \frac{dx}{\frac{1}{9}-\frac{2}{3}x-x^2}$$
$$= \frac{1}{9}\int \frac{dx}{\frac{2}{9}-(x+\frac{1}{3})^2}$$
$$= \frac{1}{9 \cdot 2 \cdot \sqrt{\frac{2}{9}}} \log \frac{\sqrt{\frac{2}{9}}+x+\frac{1}{3}}{\sqrt{\frac{2}{9}}-x-\frac{1}{3}}$$
$$= \frac{1}{6\sqrt{2}} \log \frac{\sqrt{2}+3x+1}{\sqrt{2}-3x-1}.$$

Type ii : $\int \frac{lx+m}{ax^2+bx+c}\,dx.$

If ax^2+bx+c has no rational factors, express the numerator as A (derivative of the denominator) + B and integrate each part separately. The method is outlined below :

Examples.

Ex. 1. $\int \frac{2x+3}{x^2+x+1}\,dx.$ (B.A. 47 M)

$$\frac{d}{dx}(x^2+x+1) = 2x+1.$$

Hence $$\int \frac{2x+3}{x^2+x+1}\,dx$$
$$= \int \frac{(2x+1)\,dx}{x^2+x+1} + 2\int \frac{dx}{x^2+x+1}$$
$$= \log(x^2+x+1) + 2\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$$
$$= \log(x^2+x+1) + \frac{4}{\sqrt{3}} \tan^{-1} \frac{x+\frac{1}{2}}{\sqrt{3}/2}$$
$$= \log(x^2+x+1) + \frac{4}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}}.$$

Ex. 2. $\int \frac{x+4}{6x-7-x^2}\,dx.$

Here $\frac{d}{dx}(6x-7-x^2) = -2x+6.$

$\therefore \quad x+4 = -\frac{1}{2}(-2x+6)+7$

$$\int \frac{x+4}{6x-7-x^2}\,dx$$
$$= -\frac{1}{2}\int \frac{-2x+6}{6x-7-x^2}\,dx + 7\int \frac{dx}{-7-(x^2-6x)}$$

$$= -\frac{1}{2}\log(6x - 7 - x^2) + 7\int\frac{dx}{2-(x-3)^2}$$
$$= -\frac{1}{2}\log(6x - 7 - x^2) + \frac{7}{2\sqrt{2}}\log\frac{\sqrt{2}+x-3}{\sqrt{2}-x+3}.$$

Exercises LIII.

Integrate

1. $\dfrac{1}{3+2x+x^2}$.
2. $\dfrac{1}{4+5x+x^2}$.
3. $\dfrac{3x+5}{x^2+4x+7}$. (B.A. 40 M)
4. $\dfrac{2x+3}{x^2+2x+5}$. (B.Sc. Sub. 42)
5. $\dfrac{2x+1}{x^2+21x+3}$.
6. $\dfrac{3x+1}{2x^2-x+5}$.
7. $\dfrac{2x+3}{-1+x-2x^2}$.
8. $\dfrac{5x+7}{6x^2+4x-1}$.
9. $\dfrac{3x+1}{2x^2+x+3}$.
10. $\dfrac{5x+1}{x^2-2x-35}$. (B.Sc. 35 M)
11. $\dfrac{1}{3x^2+13x-10}$.
12. $\dfrac{1}{e^x+2e^{-x}-3}$.
13. $\dfrac{x^2}{x^6+2x^3+2}$. (B.A. 45 M)
14. $\dfrac{x^3}{-x^8+2x^4+4}$.
15. $\dfrac{4x^2+3}{8x^2+4x+5}$.
16. $\dfrac{x^2+x+1}{x^2-x+1}$. (B.Sc. 41 M)
17. $\dfrac{x^2-x+1}{1-x-x^2}$.
18. $\dfrac{x^4}{x^2+x+1}$.

19. Show that $\displaystyle\int_0^{\pi/2}\frac{\sin 2\theta}{\sin^4\theta+\cos^4\theta}\,d\theta = \frac{\pi}{4}$.

20. Show that $\displaystyle\int_{-\infty}^{+\infty}\frac{dx}{a+2bx+cx^2} = \frac{\pi}{\sqrt{ac-b^2}}$.

21. Show that $\displaystyle\int_0^1\frac{dx}{1+2x\cos\theta+x^2} = \frac{\theta}{2\sin\theta}$.

§ 48·3. **Rule (c).** If the denominator can be resolved into rational factors of the first or second degree, the method of partial fractions is to be used.

Examples.

Ex. 1. $\int \frac{dx}{x^2-a^2}$.

Let $\frac{1}{x^2-a^2} = \frac{A}{x-a} + \frac{B}{x+a}$.

Then $1 \equiv A(x+a) + B(x-a)$.

Putting $x = a$, $A = \frac{1}{2a}$ and $x = -a$, $B = -\frac{1}{2a}$.

$$\therefore \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int \frac{dx}{x-a} - \frac{1}{2a}\int \frac{dx}{x+a} = \frac{1}{2a}\log\frac{x-a}{x+a}.$$

Ex. 2. Similarly $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\int \frac{dx}{a-x} + \frac{1}{2a}\int \frac{dx}{a+x}$

$$= -\frac{1}{2a}\log(a-x) + \frac{1}{2a}\log(a+x)$$

$$= \frac{1}{2a}\log\frac{a+x}{a-x}.$$

Ex. 3. $\int \frac{x^3}{(x-1)(x-2)}\,dx$. (B.A. 48 M)

[Here the degree of the numerator is higher than that of the denominator. Hence rule (*a*) is to be applied.]

By division $\frac{x^3}{(x-1)(x-2)} = x + 3 + \frac{7x-6}{(x-1)(x-2)}$.

Let $\frac{7x-6}{(x-1)(x-2)} \equiv \frac{A}{x-1} + \frac{B}{x-2}$.

$$\therefore\ 7x - 6 \equiv A(x-2) + B(x-1).$$

Putting $x = 1$ and 2 in turn, $A = -1$ and $B = 8$.

$$\therefore \int \frac{x^3}{(x-1)(x-2)}\,dx = \int \left(x + 3 - \frac{1}{x-1} + \frac{8}{x-2}\right)dx$$

$$= \frac{x^2}{2} + 3x - \log(x-1) + 8\log(x-2).$$

Ex. 4. $\int \frac{3x+1}{(x-1)^2(x+3)}\,dx$. (B.Sc. 48 M)

Assume $\frac{3x+1}{(x-1)^2(x+3)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$.

$\therefore 3x + 1 \equiv A(x-1)(x+3) + B(x+3) + C(x-1)^2$.

Put $x = 1$, $B = 1$; put $x = -3$, $C = -\frac{1}{2}$.

Put $x = 0$, $-3A + 3B + C = 1$, hence $A = \frac{1}{2}$.

$$\int \frac{3x+1}{(x-1)^2(x+3)}\,dx$$
$$= \frac{1}{2}\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} - \frac{1}{2}\int \frac{dx}{x+3}$$
$$= \frac{1}{2}\log(x-1) - \frac{1}{x-1} - \frac{1}{2}\log(x+3)$$
$$= \frac{1}{2}\log\frac{x-1}{x+3} - \frac{1}{x-1}.$$

Ex. 5. $\displaystyle\int \frac{2dx}{(1-x)(1+x^2)}.$

Let $\displaystyle\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2}.$

$\therefore\ 2 = A(1+x^2) + (Bx+C)(1-x).$

Put $x = 1$, $A = 1$; put $x = 0$, $A + C = 2$. $\therefore\ C = 1$.

Put $x = -1$, $2A + 2(-B + C) = 2$. $\therefore\ B = 1$.

Hence the integral is $\displaystyle\int \frac{dx}{1-x} + \int \frac{x+1}{x^2+1}\,dx$

$$= -\log(1-x) + \int \frac{x}{x^2+1}\,dx + \int \frac{dx}{x^2+1}$$
$$= -\log(1-x) + \tfrac{1}{2}\log(x^2+1) + \tan^{-1}x.$$

Exercises LIV.

Integrate

1. $\dfrac{1}{(x+1)(x+2)}.$

2. $\dfrac{2x+3}{(2x+1)(1-3x)}.$

3. $\dfrac{1}{2-3x+x^2}.$

4. $\dfrac{x}{(x-1)(x-2)(x-3)}.$

5. $\dfrac{x^2+11x+14}{(x+3)(x^2-4)}.$ (B.Sc. 48 M)

6. $\dfrac{x^2+1}{(x^2-1)(2x+1)}.$ (B.Sc. 40 M)

7. $\dfrac{10x-21}{(2x-3)(2x+5)}.$

8. $\dfrac{1-4x^2}{x(1-4x^2)}.$

9. $\dfrac{x^2-1}{x^2-4}.$

10. $\dfrac{x}{(x-1)^2(x+2)}.$ (B.Sc. 50 M

11. $\dfrac{x^2}{(x-1)^3(x+1)}.$ (B.Sc. 52 M

12. $\dfrac{x+1}{(x-1)^2(x+2)}.$ (B.Sc. 49 M

13. $\dfrac{3x+1}{(x+2)^2}.$ (B.Sc. Sub. 4

14. $\dfrac{x^2+x+1}{x^2(x+2)}$.

15. $\dfrac{x^2}{(x+2)^2(x+1)}$. (B.A. 39 M)

16. $\dfrac{x}{(1+x)(1+x^2)}$. (B.A. 56 M)

17. $\dfrac{1}{(x+1)^2(x^2+1)}$. (B.Sc. 50 M)

18. $\dfrac{1}{a^4-x^4}$.

19. $\dfrac{3x-1}{(x^2+1)(x-2)}$.

20. $\dfrac{5x}{(x+1)(x^2+4)}$.

21. $\dfrac{2x}{(x^2+1)(x^2+3)}$.

22. $\dfrac{x}{(x^2+a)(x^2+b)}$.

23. $\dfrac{x}{x^3-1}$. (B.A. 50 M)

24. $\dfrac{x}{x^3+1}$.

25. $\dfrac{1}{x^3+x^2+x}$.

26. $\dfrac{1}{x^2+x^3+x^4}$.

27. $\dfrac{1}{1+x^4}$. (B.Sc. 38 M)

28. $\dfrac{x^2+1}{x^4+1}$. (B.Sc. 47 M)

29. $\dfrac{x}{x^4+a^4}$.

30. $\dfrac{x^7}{x^{12}+1}$.

31. $\dfrac{x^3+x+1}{(x^2+1)^2(x+2)}$. (B.Sc. 52 M)

32. $\dfrac{x}{(x+1)(x^2+4x+13)}$.

33. $\dfrac{\cos x}{(1+\sin x)(2+\sin x)}$.

34. $\dfrac{\cos x}{\sin x(1+4\sin x)}$. (B.A. 53 M)

35. $\dfrac{\sec^2 x}{1-4\tan^2 x}$.

36. $\dfrac{1}{\sin x+\sin^2 x}$. (B.Sc. 49 M)

37. $\dfrac{1}{3\sin x+\sin 2x}$.

38. Evaluate $\displaystyle\int_1^{\infty}\frac{dx}{x^2(x+1)}$. (B.Sc. Sub. 48)

39. Evaluate $\displaystyle\int_0^1\frac{dx}{1+x^3}$. (B.Sc. 46 M)

40. Show that $\displaystyle\int_0^{\pi/2}\frac{\sin x\cos x\,dx}{\cos^2 x+3\cos x+2}=\log\frac{9}{8}$. (B.Sc. 55 M)

41. Evaluate $\displaystyle\int\frac{dx}{\cos x-\cos^2 x}$.

§ 48·4. Special cases.

(1) In certain cases a substitution materially shortens the work. This is especially so if some power of x, say, x^{n-1} is a factor of the numerator and the rest of the fraction is a rational function of x^n.

Examples.

Ex. 1. $\displaystyle\int \frac{x^2\,dx}{x^6+2x^3+2}$.

In this case since the numerator x^2 is $\frac{1}{3}$ of the d.c. of x^3 put $x^3 = t$. $\therefore\ 3x^2\,dx = dt$.

$$\therefore \int \frac{x^2\,dx}{x^6+2x^3+2} = \frac{1}{3}\int \frac{dt}{t^2+2t+2}$$
$$= \frac{1}{3}\int \frac{dt}{(t+1)^2+1}$$
$$= \tfrac{1}{3}\tan^{-1}(t+1)$$
$$= \tfrac{1}{3}\tan^{-1}(x^3+1).$$

Ex. 2. $\displaystyle\int \frac{dx}{x(x^3+1)}$

$$= \int \frac{x^2\,dx}{x^3(x^3+1)}\ ;\ \text{put } x^3 = t.\quad \therefore\ 3x^2dx = dt.$$
$$= \frac{1}{3}\int \frac{dt}{t(t+1)}$$
$$= \frac{1}{3}\int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt$$
$$= \tfrac{1}{3}\{\log t - \log(t+1)\}$$
$$= \frac{1}{3}\log\frac{t}{t+1}$$
$$= \frac{1}{3}\log\frac{x^3}{x^3+1}.$$

(2) In fractions in which there is no odd power of x and in which the denominator can be broken up into factors of the form $x^2 \pm a^2$, it is not necessary to resolve the denominator into linear factors. The partial fraction corresponding to each factor x^2+a^2 or $x^2 - a^2$ should be obtained regarding x^2 as the variable.

Examples.

Ex. 1. $\displaystyle\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$.

$$\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{A}{x^2+a^2} + \frac{B}{x^2+b^2}$$
$$= \frac{1}{a^2-b^2}\left(\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\right.$$

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$$\therefore \int \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

$$= \frac{1}{a^2-b^2}\int \frac{dx}{x^2+b^2} - \frac{1}{a^2-b^2}\int \frac{dx}{x^2+a^2}$$

$$= \frac{1}{a^2-b^2}\cdot\frac{1}{b}\tan^{-1}.\left(\frac{x}{b}\right) - \frac{1}{a^2-b^2}\cdot\frac{1}{a}\tan^{-1}.\left(\frac{x}{a}\right)$$

$$= \frac{1}{a^2-b^2}\left(\frac{1}{b}\tan^{-1}\frac{x}{b} - \frac{1}{a}\tan^{-1}\frac{x}{a}\right).$$

(3) Sometimes it is more convenient to break up the denominator completely into linear factors although this may introduce imaginary numbers. After resolution into partial fractions or after integration, the pairs of terms corresponding to conjugate roots can be combined and reduced to real form by the help of De Moivre's Theorem.

(4) Very often expressions involving x^2+a^2 can be integrated more conveniently by the substitution $x = a \tan \theta$.

Examples.

Ex. 1. $\int \frac{dx}{(1+x^2)^2}$.

Putting $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$.

$$\therefore \int \frac{dx}{(1+x^2)^2} = \int \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^2}$$

$$= \int \cos^2\theta\, d\theta$$

$$= \tfrac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta$$

$$= \frac{1}{2}\tan^{-1}x + \frac{1}{2}\cdot\frac{x}{1+x^2}.$$

Ex. 2. $\int \frac{x\, dx}{(x^2+2x+2)^2}$

$$\frac{x}{(x^2+2x+2)^2} = \frac{x}{\{(x+1)^2+1\}^2}.$$

Put $x+1 = \tan\theta$. $\therefore$ $dx = \sec^2\theta\, d\theta$.

$$\therefore \int \frac{x}{(x^2+2x+2)^2} = \int \frac{(\tan\theta - 1)\sec^2\theta\, d\theta}{(\tan^2\theta+1)^2}$$

$$= \int (\tan\theta - 1)\cos^2\theta\, d\theta$$

$$= \int (\sin\theta\cos\theta - \cos^2\theta)\, d\theta$$

$$= \int \tfrac{1}{2}\sin 2\theta\, d\theta - \int \tfrac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= -\tfrac{1}{4}\cos 2\theta - \tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta.$$

$$=\frac{1}{4}\frac{(x+1)^2-1}{(x+1)^2+1}-\frac{1}{2}\tan^{-1}(x+1)-\frac{1}{4}\frac{2(x+1)}{1+(x+1)^2}$$

$$=\frac{x^2+2x}{4(x^2+2x+2)}-\frac{(x+1)}{2(x^2+2x+2)}-\frac{1}{2}\tan^{-1}(x+1)$$

$$=\frac{x^2-2}{4(x^2+2x+2)}-\frac{1}{2}\tan^{-1}(x+1).$$

Examples LV.

Integrate

1. $\dfrac{1}{x(x^n+1)}$. (B.E. 48)
2. $\dfrac{1}{x(x^5+1)}$.
3. $\dfrac{1}{x(x^2+1)^3}$.
4. $\dfrac{1}{x(2x^7+1)}$. (T.U. 55)
5. $\dfrac{2x}{(x^2+1)(x^2+3)}$.
6. $\dfrac{x}{(x^2-a^2)(x^2-b^2)}$.
7. $\dfrac{x^2}{(x^2+1)^2}$.
8. $\dfrac{x^4}{(x^2+1)^2}$.
9. $\dfrac{1}{x^2(1+x^2)^2}$.
10. $\dfrac{x^3}{(a^2+x^2)^2}$.
11. $\dfrac{1}{(x^2+4x+5)^2}$.
12. $\dfrac{x^5}{(x^2+a^2)^2}$.
13. $\dfrac{1}{(a^2+b^2x^2)^2}$.
14. Prove that

$$\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}=\frac{\pi}{2(b+c)(c+a)(a+b)}.$$

15. Integrate $\dfrac{x^2}{(x^2-1)(x^2+2)}$.
16. $\dfrac{x^2+4}{(x^2+1)(x^2+3)}$.
17. Prove that $\displaystyle\int_0^a \frac{a^2-x^2}{(a^2+x^2)^2}\,dx=\frac{1}{2a}$.

§ 49. Integration of irrational functions.

Many irrational expressions can be rationalised by a suitable change of variable as will be explained later on.

It has already been shown that

(1) $$\int \frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}.$$

(2) $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a}.$

(3) $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a}$ or $\log\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)$.

Let us consider the allied integrals.

1. $\int \sqrt{a^2-x^2}\,dx$. Put $x = a\sin\theta$, then $dx = a\cos\theta\,d\theta$.

$$\therefore \int \sqrt{a^2-x^2}\,dx = a^2\int \cos^2\theta\,d\theta$$
$$= \frac{a^2}{2}\int (1+\cos 2\theta)\,d\theta$$
$$= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)$$
$$= \frac{a^2}{2}(\theta + \sin\theta\cos\theta)$$
$$= \frac{a^2}{2}\left\{\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right\}$$
$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{2}.$$

2. $\int \sqrt{a^2+x^2}\,dx$. Put $x = a\sinh\theta$, $dx = a\cosh\theta\,d\theta$.

$$\therefore \int \sqrt{a^2+x^2}\,dx = a^2\int \cosh^2\theta\,d\theta$$
$$= \frac{a^2}{2}\int (1+\cosh 2\theta)\,d\theta$$
$$= \frac{a^2}{2}\left(\theta + \frac{\sinh 2\theta}{2}\right)$$
$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\sinh\theta\cosh\theta$$
$$= \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{a^2}{2}\cdot\frac{x}{a}\sqrt{1+\frac{x^2}{a^2}}$$
$$= \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{x\sqrt{a^2+x^2}}{2}.$$

3. $\int \sqrt{x^2-a^2}\,dx$. Put $x = a\cosh\theta$, $dx = a\sinh\theta\,d\theta$.

$$\therefore \int \sqrt{x^2-a^2}\,dx = a^2\int \sinh^2\theta\,d\theta$$
$$= \frac{a^2}{2}\int (\cosh 2\theta - 1)\,d\theta$$
$$= \frac{a^2}{3}\left[\frac{\sinh 2\theta}{2} - \theta\right]$$
$$= \frac{a^2}{2}\sinh\theta\cosh\theta - \frac{a^2}{2}\theta$$

$$= \frac{a^2}{2}\frac{x}{a}\sqrt{\frac{x^2}{a^2}-1} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$$

$$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}.$$

Case i. Integration of the form $\dfrac{1}{\sqrt{\mathbf{ax^2+bx+c}}}$.

Divide the expression under the root by the numerical value of the coefficient of x^2 and complete the square of the terms which contain x; the integral reduces to one of the three forms above.

Examples.

Ex. 1. $\displaystyle\int \frac{dx}{\sqrt{2-3x+x^2}}$. (B.Sc. 51 M)

$$= \int \frac{dx}{\sqrt{(x-3/2)^2-\frac{1}{4}}} = \cosh^{-1}(2x-3) \text{ (by 14 § 43).}$$

Ex. 2. $\displaystyle\int \frac{dx}{\sqrt{3x-x^2-2}}$. (B.A. 50 M)

$$= \int \frac{dx}{\sqrt{\frac{1}{4}-(x-3/2)^2}} = \sin^{-1}(2x-3) \text{ (by 13 § 43).}$$

Ex. 3. $\displaystyle\int \frac{dx}{\sqrt{x(3-2x)}} = \int \frac{dx}{\sqrt{3x-2x^2}}$.

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\frac{3}{2}x-x^2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{(\frac{3}{4})^2-(x-\frac{3}{4})^2}}$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\frac{x-3/4}{3/4} = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4x-3}{3}\right).$$

Ex. 4. $\displaystyle\int \frac{dx}{\sqrt{3x^2+x-2}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{x^2+\frac{x}{3}-2/3}}$

$$= \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{(x+\frac{1}{6})^2-\frac{1}{36}-\frac{2}{3}}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{(x+\frac{1}{6})^2-(\frac{5}{6})^2}}$$

$$= \frac{1}{\sqrt{3}}\cosh^{-1}\frac{x+\frac{1}{6}}{\frac{5}{6}} = \frac{1}{\sqrt{3}}\cosh^{-1}\frac{6x+1}{5}$$

Case ii. Integration of the form $\dfrac{\mathbf{px+q}}{\sqrt{\mathbf{ax^2+bx+c}}}$.

Put the numerator equal to A (d.c. of the expression under the radical sign) $+ B$ when A and B are constants, *i.e.*, $px+q \equiv A(2ax+b)+B$. The values of A and B can easily be determined. The integral breaks up into two parts in one of which the numerator is the differential coefficient of ax^2+bx+c and in the

other the numerator does not involve x. The method of integration of the two parts is illustrated by the following examples.

Examples.

Ex. 1. $\displaystyle\int \frac{x}{\sqrt{x^2+x+1}}\,dx.$

Let us assume that $x = A$ (d.c. of $x^2+x+1) + B$

$$= A(2x+1) + B.$$

$$\therefore \quad A = \tfrac{1}{2},\ B = -\tfrac{1}{2}.$$

$$\therefore \int \frac{x}{\sqrt{x^2+x+1}}\,dx = \int \frac{\frac{1}{2}(2x+1) - \frac{1}{2}}{\sqrt{x^2+x+1}}\,dx$$

$$= \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+x+1}}$$

$$= \sqrt{x^2+x+1} - \frac{1}{2}\int \frac{dx}{\sqrt{(x+\frac{1}{2})^2 + 3/4}}$$

$$= \sqrt{x^2+x+1} - \frac{1}{2}\sinh^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \sqrt{x^2+x+1} - \frac{1}{2}\sinh^{-1}\frac{2x+1}{\sqrt{3}}.$$

Ex. 2. $\displaystyle\int \frac{6x+5}{\sqrt{6+x-2x^2}}\,dx.$

Let $6x+5 = A$ (d.c. of $6+x-2x^2) + B$

$$= A(1-4x) + B.$$

$$\therefore \quad A = -3/2\ ;\ B = 13/2.$$

$$\therefore \int \frac{6x+5}{\sqrt{6+x-2x^2}}\,dx = \int \frac{-\frac{3}{2}(1-4x) + \frac{13}{2}}{\sqrt{6+x-2x^2}}\,dx$$

$$= -\frac{3}{2}\int \frac{1-4x}{\sqrt{6+x-2x^2}}\,dx + \frac{13}{2}\int \frac{dx}{\sqrt{6+x-2x^2}}$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\int \frac{dx}{\sqrt{3+\frac{1}{2}x-x^2}}$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\int \frac{dx}{\sqrt{\frac{49}{16}-(x-\frac{1}{4})^2}}$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}} \sin^{-1} \frac{x-\frac{1}{4}}{\frac{7}{4}}$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}} \sin^{-1} \frac{4x-1}{7}.$$

Ex. 3. $\displaystyle\int \frac{3x-2}{\sqrt{4x^2-4x-5}}\,dx.$

$3x - 2 = A(8x-4) + B.$

$\therefore \quad A = 3/8, B = -1/2.$

$$\therefore \int \frac{3x-2}{\sqrt{4x^2-4x-5}}\,dx = \int \frac{3(8x-4)}{8\sqrt{4x^2-4x-5}}\,dx - \frac{1}{2}\int \frac{dx}{\sqrt{4x^2-4x-5}}$$

$$= \tfrac{3}{4}\sqrt{4x^2-4x-5} - \tfrac{1}{4}\int \frac{dx}{\sqrt{(x-\frac{1}{2})^2-\frac{3}{2}}}$$

$$= \tfrac{3}{4}\sqrt{4x^2-4x-5} - \tfrac{1}{4}\cosh^{-1}\frac{2x-1}{\sqrt{6}}.$$

Ex. 4. $$\int \sqrt{\frac{5-x}{2-x}}\,dx = \int \frac{5-x}{\sqrt{(x-2)(5-x)}}\,dx$$

$$= \int \frac{5-x}{\sqrt{-10+7x-x^2}}\,dx$$

which can be easily found as

$$\sqrt{10-7x-x^2} + \tfrac{7}{2}\sin^{-1}\left(\frac{2x-7}{3}\right).$$

Exercises LVI.

Integrate

1. $\dfrac{1}{\sqrt{2x-x^2}}.$ (B.Sc. Sub. 46)

2. $\dfrac{2x}{\sqrt{3+4x+x^2}}.$ (B.Sc. 43 M)

3. $\dfrac{2x-3}{\sqrt{2x^2-7x+5}}.$

4. $\dfrac{1-4x}{\sqrt{x^2-2x+4}}.$

5. $\dfrac{x+1}{\sqrt{x(x-2)}}.$

6. $\dfrac{x+1}{\sqrt{2x^2+x-3}}.$

7. $\dfrac{3x-4}{\sqrt{3x^2+4x+7}}.$

8. $\sqrt{\dfrac{x-1}{2x-3}}$.

[*Hint.* Multiply Nr. and Dr. by $\sqrt{x-1}$.]

9. $\sqrt{\dfrac{3-2x}{1-x}}$.

10. $\dfrac{1}{\sqrt{8+3x-x^2}}$.

11. $\dfrac{2x-1}{\sqrt{x^2+5x+6}}$.

12. $\dfrac{1}{\sqrt{(x-\alpha)(\beta-x)}}$.

13. $\dfrac{2+x}{\sqrt{x^2-1}}$.

14. $\dfrac{1}{\sqrt{(a-x)(b+x)}}$.

15. $\dfrac{2x-4}{\sqrt{3x^2+4x+7}}$.

16. $\dfrac{2x}{\sqrt{6-5x^2-x^4}}$ (put $x^2=t$).

17. $\dfrac{\cos x}{\sqrt{4-\sin^2 x}}$.

18. $\dfrac{\sec^2 x}{\sqrt{9+\tan^2 x}}$.

19. $\dfrac{x+2}{\sqrt{x^2+x+1}}$. (B.Sc. Anc. 61)

20. Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{3x+7}{\sqrt{1-x-x^2}}\,dx$.

Case iii. Integration of $\sqrt{\mathbf{ax^2+bx+c}}$ and $(\mathbf{px+q})\sqrt{\mathbf{ax^2+bx+c}}$.

It has already been shown that

(1) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\frac{x}{a}$.

(2) $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2}+\frac{1}{2}a^2\sinh^{-1}\left(\frac{x}{a}\right)$.

(3) $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2}-\frac{1}{2}a^2\cosh^{-1}\left(\frac{x}{a}\right)$.

The integration of $\sqrt{ax^2+bx+c}$ can be accomplished by reducing ax^2+bx+c to the form $a\{(x+\alpha)^2\pm\beta^2\}$.

Examples.

Ex. 1. $\int\sqrt{x^2+2x+10}\,dx = \int\sqrt{(x+1)^2+9}\,dx$

$$=\frac{1}{2}(x+1)\sqrt{x^2+2x+10}+\frac{9}{2}\sinh^{-1}\left(\frac{x+1}{3}\right).$$

Ex. 2. $\int\sqrt{1+x-2x^2}\,dx = \sqrt{2}\displaystyle\int\sqrt{\frac{1}{2}+\frac{x}{2}-x^2}\,dx$

$$=\sqrt{2}\int\sqrt{\frac{9}{16}-\left(x-\frac{1}{4}\right)^2}\,dx$$

$$= \sqrt{2}\int \sqrt{\left(\frac{3}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2}\, dx$$

$$= \sqrt{2}\left(x - \frac{1}{4}\right)\sqrt{\frac{9}{16} - \left(x - \frac{1}{4}\right)^2}$$

$$+ \frac{1}{2} \times \frac{9}{16}\sin^{-1}\left(\frac{x - \frac{1}{4}}{\frac{3}{4}}\right)$$

$$= \frac{4x - 1}{4}\sqrt{1 + x - 2x^2} + \frac{9}{32}\sin^{-1}\frac{4x - 1}{3}.$$

Ex. 3. $\int (3x - 2)\sqrt{x^2 + x + 1}\, dx.$

Let us assume that $3x - 2 = A(2x + 1) + B.$

$$\therefore\quad A = \frac{3}{2}, B = -\frac{7}{2}.$$

$$\therefore\quad \int (3x - 2)\sqrt{x^2 + x + 1}\, dx$$

$$= \int \{\tfrac{3}{2}(2x + 1) - \tfrac{7}{2}\}\sqrt{x^2 + x + 1}\, dx$$

$$= \tfrac{3}{2}\int (2x + 1)\sqrt{x^2 + x + 1}\, dx - \tfrac{7}{2}\int \sqrt{x^2 + x + 1}\, dx$$

$$= (x^2 + x + 1)^{3/2} - \tfrac{7}{2}\int \sqrt{(x + \tfrac{1}{2})^2 + \tfrac{3}{4}}\, dx$$

$$= (x^2 + x + 1)^{3/2} - \tfrac{7}{4}(x + \tfrac{1}{2})\sqrt{x^2 + x + 1}$$

$$- \tfrac{7}{4}\cdot\tfrac{3}{4}\sinh^{-1}\frac{x + \frac{1}{2}}{\sqrt{3}/2}$$

$$= (x^2 + x + 1)^{3/2} - \frac{7(2x + 1)}{8}\sqrt{x^2 + x + 1}$$

$$- \frac{21}{16}\sin^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right).$$

Case iv. In some cases it is more convenient to proceed as below :

An algebraical expression involving only one irrational quantity $\sqrt{ax + b}$ can be rationalised by the substitution $ax + b = t^2$ as in the following examples and then its integral can be found by the methods already studied.

Examples.

Ex. 1. $\displaystyle\int \frac{x^2}{\sqrt{x + 5}}\, dx.$ Put $x + 5 = t^2$

$\therefore\ dx = 2t\, dt.$

$$\therefore\quad \int \frac{x^2}{\sqrt{x + 5}}\, dx = \int \frac{(t^2 - 5)^2\, 2t\, dt}{t}$$

$$= 2\int (t^2 - 5)^2\, dt$$

$$= 2\int (t^4 - 10t^2 + 25)\, dt$$
$$= 2\left(\frac{t^5}{5} - 10\frac{t^3}{3} + 25t\right)$$
$$= \frac{2t}{15}(3t^4 - 50t^2 + 375)$$
$$= \frac{2\sqrt{x+5}}{15}\{3(x+5)^2 - 50(x+5) + 375\}.$$

Ex. 2. $\int \frac{\sqrt{x}}{1+x}\, dx.$ Put $x = t^2$; $dx = 2t\, dt$.

$$\therefore \int \frac{\sqrt{x}}{1+x}\, dx = \int \frac{t\,.\,2t\, dt}{1+t^2}$$
$$= 2\int \frac{t^2+1-1}{1+t^2}\, dt$$
$$= 2\int \left(1 - \frac{1}{1+t^2}\right) dt$$
$$= 2t - 2\tan^{-1} t$$
$$= 2\sqrt{x} - 2\tan^{-1}(\sqrt{x}).$$

Ex. 3. $\int \frac{dx}{(1+x)^{3/2} + (1+x)^{1/2}}.$ Put $1 + x = t^2$, $dx = 2t\, dt$.

$$\therefore \int \frac{dx}{(1+x)^{3/2} + (1+x)^{1/2}} = \int \frac{2t\, dt}{t^3+t} = 2\int \frac{dt}{t^2+1}$$
$$= 2\tan^{-1}(t) = 2\tan^{-1}\sqrt{1+x}.$$

Ex. 4. $\int \frac{dx}{1+\sqrt[3]{x+a}}.$ Put $x + a = t^3$, $dx = 3t^2\, dt$.

$$= \int \frac{3t^2\, dt}{1+t} = \tfrac{3}{2}t^2 - 3t + 3\log(1+t)$$
$$= \tfrac{3}{2}(x+a)^{2/3} - 3(x+a)^{1/3} + 3\log(1 + \sqrt[3]{x+a}).$$

Case v. Any expression of the form

$$\frac{1}{(x-k)\sqrt{ax^2+bx+c}}$$

can be integrated by the substitution $x - k = \frac{1}{t}$, the expression is thereby reduced to the form $\frac{1}{\sqrt{Ax^2+Bx+C}}$ which has been already considered.

Examples.

Ex. 1. $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}.$ (B.Sc. 52 M)

Put $x+1=\dfrac{1}{t}$; $dx=-\dfrac{1}{t^2}\,dt.$

$$\therefore \int \frac{dx}{(x+1)\sqrt{x^2+x+1}}=\int \frac{-\,dt/t^2}{\dfrac{1}{t}\sqrt{\dfrac{1}{t^2}-\dfrac{1}{t}+1}}$$

$$=-\int \frac{dt}{\sqrt{1-t+t^2}}=-\int \frac{dt}{\sqrt{\dfrac{3}{4}+\left(t-\dfrac{1}{2}\right)^2}}$$

$$=-\sinh^{-1}\frac{t-\frac{1}{2}}{\sqrt{3}/2}$$

$$=-\sinh^{-1}\frac{1-x}{\sqrt{3}\,(1+x)}.$$

Ex. 2. $\displaystyle\int \frac{dx}{(3+x)\sqrt{x}}$. Put $3+x=\dfrac{1}{t}$.

$$\therefore\ dx=-\frac{1}{t^2}\,dt.$$

$$=\int \frac{-\dfrac{1}{t^2}\,dt}{\dfrac{1}{t}\sqrt{\left(\dfrac{1}{t}-3\right)}}=-\int \frac{dt}{\sqrt{t-3t^2}}$$

$$=-\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\dfrac{t}{3}-t^2}}=-\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(\dfrac{1}{6}\right)^2-\left(t-\dfrac{1}{6}\right)^2}}$$

$$=-\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{t-\frac{1}{6}}{\frac{1}{6}}\right)=-\frac{1}{\sqrt{3}}\sin^{-1}(6t-1)$$

$$=-\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{3-x}{3+x}\right).$$

Case vi. Integration of $\dfrac{1}{(Ax^2+B)\sqrt{Cx^2+D}}$.

To integrate this we have to put $x=\dfrac{1}{t}$. This substitution will facilitate integration.

Example. $\displaystyle\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$.

Putting $x = \frac{1}{t}$ and $dx = -\frac{1}{t^2}\,dt$; we have

$$\int \frac{dx}{(1+x^2)\sqrt{1-x^2}} = \int \frac{-dt/t^2}{\left(1+\frac{1}{t^2}\right)\sqrt{1-\frac{1}{t^2}}}$$

$$= -\int \frac{t\,dt}{(t^2+1)\sqrt{t^2-1}}$$

$$= -\int \frac{u\,du}{(u^2+2)\,u} \text{ on putting } t^2-1=u^2,\quad t\,dt = u\,du$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{t^2-1}}{\sqrt{2}}$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{1-x^2}}{x\sqrt{2}}.$$

Exercises LVII. Integrate

1. $\sqrt{x^2+2x+5}.$
2. $\sqrt{x^2+2x-3}.$
3. $\sqrt{3-2x-x^2}.$
4. $\dfrac{1}{(x+1)\sqrt{1-x^2}}.$
5. $\dfrac{1}{(x-1)\sqrt{1-x^2}}.$
6. $\dfrac{x^2}{\sqrt{x+2}}.$
7. $\dfrac{1}{(x+1)\sqrt{x^2-1}}.$
8. $\dfrac{1}{(x+1)\sqrt{x^2+1}}.$
9. $\dfrac{1}{3+\sqrt{x}}.$
10. $\dfrac{1}{(x+2)\sqrt{x+3}}.$
11. $x^3\sqrt{x^2+4}.$
12. $\dfrac{1}{x\sqrt{x-1}}.$
13. $\dfrac{x^2}{\sqrt{x+5}}.$
14. $\dfrac{1}{(x-1)\sqrt{x^2+2x-8}}.$
15. $\dfrac{1}{(x+b)\sqrt{x+a}}.$
16. $\dfrac{x^2+1}{\sqrt{x+1}}.$
17. $\dfrac{\sqrt{x}}{a+\sqrt{x}}.$
18. $\dfrac{1}{(1-x)\sqrt{x}}.$
19. $\dfrac{1}{x\sqrt{x^2+6x+109}}.$
20. $\dfrac{1}{x\sqrt{x^2+x+1}}.$
21. $\dfrac{1}{(x-1)\sqrt{x^2-2x+3}}.$
22. $\dfrac{1}{(2x+3)\sqrt{x+5}}.$
23. $\dfrac{1}{(1-2x)\sqrt{3x^2+2x+1}}.$

24. Show that $\displaystyle\int_1^2 \frac{dx}{(x+1)\sqrt{x^2-1}} = \frac{1}{\sqrt{3}}$.

25. Show that $\displaystyle\int_0^1 \frac{dx}{(1+x^2)\sqrt{x^2+2}} = \frac{\pi}{6}$.

26. Show that $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{(1-2x^2)\sqrt{1-x^2}} = \frac{1}{2}\log(2+\sqrt{3})$.

(B.Sc. 51 M)

27. $\displaystyle\int_8^{15} \frac{dx}{(x-3)\sqrt{x+1}} = \frac{1}{2}\log\frac{5}{3}$. (B.A. 54 M)

28. $\displaystyle\int \frac{dx}{x\sqrt{5x^2+2x+1}}$. (B.E. 55)

29. $\displaystyle\int_0^1 \frac{dx}{(2-x^2)\sqrt{1-x^2}}$. (B.A. 50 M)

30. $\displaystyle\int \frac{dx}{(x-1)\sqrt{x^2+2}}$. (B.Sc. 60)

Case vii. Many algebraical functions which involve the square root of a quadratic expression can be rationalised by a trigonometrical substitution and their integration is often thereby simplified.

If an expression involves the irrational quantity $\sqrt{a^2-x^2}$ or $\sqrt{a^2+x^2}$ or $\sqrt{x^2-a^2}$ and no other radical, we can put $x = a\sin\theta$, or $a\tan\theta$, or $a\sec\theta$ respectively in the above cases and we shall get rid of the square root.

Examples.

Ex. 1. $\displaystyle\int \frac{x^3+1}{\sqrt{1-x^2}}\,dx$ [put $x = \sin\theta$; $dx = \cos\theta\,d\theta$.]

$$= \int \frac{\sin^3\theta + 1}{\cos\theta}\cos\theta\,d\theta = \int(1+\sin^3\theta)\,d\theta$$

$$= \int\left\{1 + \frac{1}{4}(3\sin\theta - \sin 3\theta)\right\}d\theta$$

(using $\sin 3\theta = 3\sin\theta - 4\sin^3$

$$= \theta - \frac{3}{4}\cos\theta + \frac{1}{12}\cos 3\theta$$

$$= \theta - \frac{3}{4}\cos\theta + \frac{1}{12}(4\cos^3\theta - 3\cos\theta)$$

$$= \theta - \cos\theta + \frac{1}{3}\cos^3\theta$$

$$= \sin^{-1}x - \sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2}.$$

Ex. 2. $\int \frac{dx}{(a^2+x^2)^{3/2}}$. (B.Sc. Sub. 45)

[Put $x = a\tan\theta$; $dx = a\sec^2\theta\, d\theta$.]

The integral $= \int \frac{a\sec^2\theta\, d\theta}{(a^2+a^2\tan^2\theta)^{3/2}} = \frac{1}{a^2}\int \frac{d\theta}{\sec\theta}$

$$= \frac{1}{a^2}\sin\theta = \frac{x}{a^2\sqrt{a^2+x^2}}.$$

Ex. 3. $\int \frac{dx}{x^2\sqrt{4+x^2}}$. [Put $x = 2\tan\theta$; $dx = 2\sec^2\theta\, d\theta$.]

$$= \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta\sqrt{4+4\tan^2\theta}}$$

$$= \int \frac{\cos\theta}{4\sin^2\theta}d\theta = -\frac{1}{4}\cdot\frac{1}{\sin\theta} = -\frac{\sqrt{x^2+4}}{4x}.$$

Ex. 4. $\int \frac{dx}{x^3\sqrt{x^2-9}}$.

[Put $x = 3\sec\theta$; $dx = 3\sec\theta\tan\theta\, d\theta$.]

Integral $= \int \frac{3\sec\theta\tan\theta\, d\theta}{27\sec^3\theta\sqrt{9\sec^2\theta-9}} = \int \frac{d\theta}{27\sec^2\theta}$

$$= \frac{1}{27}\int \cos^2\theta\, d\theta$$

$$= \frac{\theta}{54} + \frac{1}{108}\sin 2\theta = \frac{\theta}{54} + \frac{\sin\theta\cos\theta}{54}$$

$$= \frac{1}{54}\sec^{-1}\left(\frac{x}{3}\right) + \frac{1}{18}\frac{\sqrt{x^2-9}}{x^2}.$$

Case viii. The expressions

$\sqrt{(x-a)(\beta-x)}$, $\frac{1}{\sqrt{(x-a)(\beta-x)}}$ and $\sqrt{\frac{x-a}{\beta-x}}$

where $\beta > a$ are all rationalised by the substitution $x = a\cos^2\theta + \beta\sin^2\theta$.

Examples.

Ex. 1. $\int \sqrt{(x-3)(7-x)}\, dx.$

Put $x = 3\cos^2\theta + 7\sin^2\theta$;

$$dx = (-6\cos\theta\sin\theta + 14\sin\theta\cos\theta)\, d\theta$$
$$= 8\sin\theta\cos\theta\, d\theta.$$

$$x - 3 = 3\cos^2\theta + 7\sin^2\theta - 3$$
$$= 7\sin^2\theta - 3\sin^2\theta = 4\sin^2\theta.$$
$$7 - x = 7 - 3\cos^2\theta - 7\sin^2\theta = 4\cos^2\theta.$$

$$\therefore \int \sqrt{(x-3)(7-x)}\, dx = \int \sqrt{4\sin^2\theta\, 4\cos^2\theta} \times 8\sin\theta\cos\theta\, d\theta$$

$$= 32\int \sin^2\theta\cos^2\theta\, d\theta = 8\int \sin^2 2\theta\, d\theta$$
$$= 4\int (1 - \cos 4\theta)\, d\theta = 4\theta - \sin 4\theta$$
$$= 4\theta - 2\sin 2\theta \,.\, \cos 2\theta$$
$$= 4\theta - 4\sin\theta\cos\theta\,(2\cos^2\theta - 1)$$
$$= 4\sin^{-1}\sqrt{\frac{x-3}{4}} - \sqrt{(x-3)(7-x)}\left\{\frac{7-x}{2} - 1\right\}$$
$$= 4\sin^{-1}\left(\frac{1}{2}\sqrt{x-3}\right) - \frac{5-x}{2}\sqrt{(x-3)(7-x)}.$$

Ex. 2. $\int \sqrt{\frac{5-x}{x-2}}\, dx.$ Put $x = 2\sin^2\theta + 5\cos^2\theta$

$$dx = -6\sin\theta\cos\theta\, d\theta.$$

$$\text{Integral} = \int \sqrt{\frac{3\sin^2\theta}{8\cos^2\theta}}\,(-6\sin\theta\cos\theta)\, d\theta$$
$$= -6\int \sin^2\theta\, d\theta = -3\int (1 - \cos 2\theta)\, d\theta$$
$$= -3\left[\theta - \frac{\sin 2\theta}{2}\right] = -3\theta + 3\sin\theta\cos\theta$$
$$= 3\sin^{-1}\left(\frac{1}{3}\sqrt{5-x}\right) + \sqrt{(5-x)(x-2)}.$$

Ex. 3. Evaluate $\int \frac{dx}{\sqrt{(x-a)(\beta-x)}}$

($\beta > a$). [*Vide* Qn. 12, Exercises LVI]

Putting $x = a\sin^2\theta + \beta\cos^2\theta$;

$$dx = 2(a - \beta)\sin\theta\cos\theta\, d\theta.$$
$$x - a = (\beta - a)\cos^2\theta \text{ and } \beta - x = (\beta - a)\sin^2\theta.$$

The integral reduces to $-2\int d\theta = -2\theta$

$$= -2\sin^{-1}\sqrt{\frac{\beta-x}{\beta-a}}$$

In particular, $\displaystyle\int_a^\beta \frac{dx}{\sqrt{(x-a)(\beta-x)}}$

$$= -2\left[\sin^{-1}\sqrt{\frac{\beta-x}{\beta-a}}\right]_a^\beta = \pi.$$

Case ix. Sometimes rationalisation of the denominator may aid integration.

Examples.

Ex. 1. $\displaystyle\int \frac{dx}{x+\sqrt{x^2-1}} = \int \{x - \sqrt{x^2-1}\}\,dx.$

$$= \tfrac{1}{2}x^2 - \int \sqrt{x^2-1}\,dx$$
$$= \tfrac{1}{2}x^2 - \tfrac{1}{2}x\sqrt{x^2-1} + \tfrac{1}{2}\cosh^{-1}x.$$

Ex. 2. $\displaystyle\int \frac{dx}{\sqrt{x}+\sqrt{1+x}} = \int (\sqrt{1+x} - \sqrt{x})\,dx$

$$= \tfrac{2}{3}(1+x)^{3/2} - \tfrac{2}{3}x^{3/2}.$$

Exercises LVIII. Integrate

1. $\dfrac{1}{x^2\sqrt{1-x^2}}$.
2. $\dfrac{1}{x^2\sqrt{1+x^2}}$.
3. $\dfrac{1}{x^2\sqrt{a^2-x^2}}$.
4. $\dfrac{x^3}{\sqrt{a^2-x^2}}$.
5. $\dfrac{x^2}{\sqrt{x^2+1}}$.
6. $\dfrac{x^2}{(1+x^2)^{3/2}}$.
7. $\dfrac{x}{(x^2+1)^{3/2}}$.
8. $\sqrt{(x+1)(4-x)}$.
9. $\sqrt{\dfrac{x-1}{2-x}}$.
10. $\dfrac{\sqrt{a^2-x^2}}{x^4}$.
11. $x\sqrt{\dfrac{1-x}{1+x}}$.
12. $\sqrt{\dfrac{x+1}{x-1}}$.
13. $\dfrac{1}{x\sqrt{1+x^6}}$.
14. $\dfrac{1}{(x+1)\sqrt{x^2-3}}$.
15. $\sqrt{\dfrac{a+x}{a-x}}$.
16. $\dfrac{1}{\sqrt{ax+b}-\sqrt{ax+c}}$.
17. Evaluate $\displaystyle\int_2^3 \frac{dx}{\sqrt{5x-6-x^2}}$.

18. Evaluate $\int_a^b \frac{x}{\sqrt{(x-a)(b-x)}}\,dx$. $(a < b)$. (B.A. Sub. 41)

19. $\int_a^b \frac{dx}{x\sqrt{(x-a)(b-x)}} = \frac{\pi}{\sqrt{ab}}$. (B.Sc. 53 T.U.)

20. Show that $\int_0^1 \sqrt{\frac{x}{1-x}}\,dx = \pi/2$.

Integrate

21. $\frac{x^2}{\sqrt{a^2-x^2}}$. (B.A. 53 M)

22. $\frac{x^3}{(8+x^2)^{3/2}}$. (B.A. 52 S)

23. $\frac{x^3+x}{\sqrt{1-x^2}}$. (B.A. 39 M)

24. $\frac{x^2}{\sqrt{1-x^2}}$.

25. $\frac{1}{(a^2-x^2)^{3/2}}$.

26. $\frac{\sqrt{1-x^2}}{1-x}$.

27. $\sqrt{\frac{a+x}{x}}$.

28. $\sqrt{\frac{a-x}{x}}$.

29. $\int_1^\infty \frac{dx}{x\sqrt{1+x^2}}$. (B.A. 51 M)

30. $\int_0^4 \frac{dx}{\sqrt{x}(1+x)}$. (B.A. 51 M)

31. $\int_0^a x\sqrt{\frac{a^2-x^2}{a^2+x^2}}\,dx$.

32. $\int_0^2 \sqrt{\frac{2+x}{2-x}}\,dx$.

33. $\int_0^a x\sqrt{\frac{a+x}{a-x}}\,dx$.

34. $\frac{1}{(x^2-a^2)^{3/2}}$. (B.Sc. 55 M)

35. Prove that $\int_2^3 \sqrt{(x-2)(3-x)}\,dx = \pi/8$. (T.U. 55)

36. Evaluate $\int_2^3 [(x-2)(3-x)]^{3/2}\,dx$. (T.U. 56)

Integrate

37. $\frac{1}{x^3(x^2-1)^{1/2}}$.

38. $\frac{x^2}{(1-x^2)^{5/2}}$.

39. $\int_a^b \sqrt{\frac{b-x}{x-a}}\,dx$. (B.E. 52)

40. $\frac{x+1}{\sqrt{1+x^2}}$.

41. $\dfrac{\sqrt{x^2 - a^2}}{x}$. 42. $\dfrac{1}{(1-x)\sqrt{1-x^2}}$.

43. $\sqrt{e^x - 1}$. (B.A. 56 M) 44. $\displaystyle\int_0^{1/\sqrt{2}} \frac{x^2}{(1-x^2)^{1/2}}\,dx$.

45. $\displaystyle\int \frac{\sin x}{\sin 4x}\,dx$ and $\displaystyle\int \frac{\sqrt{1+x^2}}{1-x^2}\,dx$. (B.Sc. Comp. Math. 59)

§ 50. Type $\displaystyle\int \frac{dx}{a + b\cos x}$.

Put $t = \tan\dfrac{x}{2}$; $dt = \dfrac{1}{2}\sec^2\dfrac{x}{2}\,dx = \dfrac{1}{2}\left(1 + \tan^2\dfrac{x}{2}\right)dx$

i.e., $dx = \dfrac{2dt}{1+t^2}$; $\cos x = \dfrac{1-t^2}{1+t^2}$.

Let $I = \displaystyle\int \frac{dx}{a + b\cos x} = \int \frac{2dt}{(a+b) + (a-b)\,t^2}$.

Two cases arise.

Case i. Let $a > b$.

$$I = \frac{2}{a-b}\int \frac{dt}{\frac{a+b}{a-b} + t^2} = \frac{2}{\sqrt{\frac{(a+b)}{(a-b)}}\,(a-b)} \times$$

$$\int \tan^{-1} t\left(\sqrt{\frac{a-b}{a+b}}\right) \qquad \text{(by § 47·2)}$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right).$$

Case ii. Let $a < b$.

$$I = 2\int \frac{dt}{a + b - (b-a)\,t^2} = \frac{2}{(b-a)}\int \frac{dt}{\frac{a+b}{b-a} - t^2}$$

$$= \frac{2}{2\,(b-a)}\sqrt{\frac{b-a}{b+a}}\log\frac{t - \sqrt{\frac{b+a}{b-a}}}{t + \sqrt{\frac{a+b}{b-a}}} \qquad \text{(by § 47·2)}$$

$$= \frac{1}{\sqrt{b^2-a^2}}\log\frac{\sqrt{b-a}\tan\frac{x}{2} - \sqrt{b+a}}{\sqrt{b-a}\tan\frac{x}{2} + \sqrt{b+a}}.$$

[*Note.*—The above substitution can be used when the denominator of the integrand is of the first degree in $\cos x$ and $\sin x$.]

Examples.

Ex. 1. Evaluate $\displaystyle\int_0^{\pi} \frac{dx}{5+4\cos x}$.

Putting $t = \tan\frac{x}{2}$ the integral reduces to

$$\int_0^{\infty} \frac{2dt}{9+t^2} = \frac{2}{3}\left\{\tan^{-1}\left(\frac{t}{3}\right)\right\}_0^{\infty} = \frac{\pi}{3}.$$

(The limits of the definite integral must be changed when the old variable x is changed to t. When $x = 0$, $t = 0$ and $x = \pi$, $t \to \infty$.)

Ex. 2. Evaluate $\displaystyle\int \frac{dx}{a\cos x + b\sin x + c}$.

Let $a = r\cos\alpha$ and $b = r\sin\alpha$.

The auxiliary constants r and α are thus given by $r = \sqrt{a^2+b^2}$ and $\alpha = \tan^{-1}\frac{b}{a}$. Hence the integral becomes

$$\int \frac{dx}{r\cos(x-\alpha)+c} = \int \frac{dy}{r\cos y + c} \text{ where } y = x - \alpha.$$

This reduces to the type considered.

Ex. 3. $\displaystyle\int_0^{\pi/2} \frac{dx}{9\cos x + 12\sin x}$. (B.Sc. 40 M)

Putting $t = \tan\frac{x}{2}$ and noting that $\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$, the integral reduces to $\displaystyle\frac{2}{3}\int_0^1 \frac{dt}{3+8t-3t^2}$ as the limits for t change to 0 and 1 when x takes the values 0 and $\pi/2$. Hence the integral is

$$\frac{2}{3}\int_0^1 \frac{dt}{(3-t)(3t+1)} = \frac{1}{15}\int_0^1 \left\{\frac{3}{3t+1} + \frac{1}{3-t}\right\} dt$$

$$= \frac{1}{15}\left\{\log\frac{3t+1}{3-t}\right\}_0^1 = \frac{1}{15}(\log 2 - \log \tfrac{1}{3}) = \frac{\log 6}{15}.$$

Exercises LIX. Evaluate

1. $\displaystyle\int \frac{dx}{4+5\cos x}$. (B.A. 51 M)

2. $\displaystyle\int \frac{dx}{13+12\cos x}$.

3. $\displaystyle\int \frac{dx}{12+13\cos x}$.

4. $\displaystyle\int_0^{\pi} \frac{dx}{13+5\cos x}$. (B.A. 36 M)

5. $\displaystyle\int \frac{dx}{4+5\sin x}$.

6. $\displaystyle\int_0^{\pi/4} \frac{d\theta}{1+\cos a\sin 2\theta}$ $(0<a<\pi/2)$.

7. $\displaystyle\int \frac{dx}{1+e\cos x}\ (e \lesseqgtr 1)$.

8. $\displaystyle\int \frac{dx}{3-4\cos x}$.

9. $\displaystyle\int \frac{dx}{2\cos x+3\sin x}$. (B.Sc. 44 M)

10. $\displaystyle\int \frac{dx}{\sin x+\sqrt{3}\cos x}$.

11. $\displaystyle\int \frac{dx}{1+3\sin x+4\cos x}$. (B.A. 55 M)

12. $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$.

13. Show that (*a*) $\displaystyle\int_0^{\pi} \frac{d\theta}{5+3\cos\theta}=\frac{\pi}{4}$.

(*b*) $\displaystyle\int_0^{\pi/2} \frac{d\theta}{1+2\cos\theta}=\frac{1}{\sqrt{3}}\log(2+\sqrt{3})$.

(*c*) $\displaystyle\int_0^{\pi} \frac{dx}{a^2-2ab\cos x+b^2}$

$$=\frac{\pi}{a^2-b^2},\ (a>b>0).$$

14. By means of the substitution $\tan\dfrac{x}{2}=\sqrt{\dfrac{1+e}{1-e}}\tan\dfrac{u}{2}$, evaluate $\displaystyle\int \frac{dx}{(1+e\cos x)^2}$. (B.A. 38 M)

15. $\displaystyle\int \frac{d\theta}{\sin\theta(1+\sin\theta)}$. (B.A. Sub. 55)

16. $\displaystyle\int_0^{\pi/2} \frac{dx}{5+4\sin x}$. (B.Sc. 59)

§ 51. Evaluate $\int \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$.

Multiplying numerator and denominator by $\sec^2 x$, the integral reduces to $\int \frac{\sec^2 x \, dx}{a^2 + b^2 \tan^2 x}$

$$= \int \frac{dt}{a^2 + b^2 t^2} \text{ on putting } \tan x = t \text{ ; } \sec^2 x \, dx = dt$$

$$= \frac{1}{ab} \tan^{-1}\left(\frac{bt}{a}\right) = \frac{1}{ab} \tan^{-1}\left(\frac{b \tan x}{a}\right).$$

Exercises LX. Integrate

1. $\frac{1}{1 + 7 \cos^2 x}$. (B.Sc. 43 M)

2. $\frac{1}{2 \sin^2 x + 3 \cos^2 x}$.

3. Show that $\int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{ab}$. (B.A. 37 M)

4. Show that $\int_{\pi/4}^{3\pi/4} \frac{d\theta}{2 \cos^2 \theta + 1} = \int_{\pi/4}^{3\pi/4} \frac{\sec^2 \theta \, d\theta}{2 + \sec^2 \theta} = \frac{2\pi}{3\sqrt{3}}$.

5. Show that $\int_0^{\pi/2} \frac{\sec^2 x \, dx}{(\sec x + \tan x)^n} = \frac{n}{n^2 - 1}$ (n being positive and greater than 1). [*Hint.* Put $z = \sec x + \tan x$.] (B.A. 38 M)

6. Show that $\int_0^{\pi/2} \frac{dx}{1 + a^2 \cos^2 x + b^2 \sin^2 x}$

$$= \frac{\pi}{2} \frac{1}{\sqrt{(1 + a^2)(1 + b^2)}}$$

7. Show that

$$\int \frac{dx}{1 - \sin^4 x} = \frac{1}{2} \tan x + \frac{1}{2\sqrt{2}} \tan^{-1} (\sqrt{2} \tan x).$$

§ 52. **Properties of definite integrals.**

1. $\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$. This is obvious from the definition of a definite integral.

2. $\int_a^b f(x)\,dx = \int_a^c (x)\,dx + \int_c^b f(x)\,dx$ where c is some value of x between a and b.

Let $\int f(x) = F(x)$.

Then $\int_a^b f(x)\,dx = F(b) - F(a)$.

The R.H.S. $= F(c) - F(a) + F(b) - F(c)$

$= F(b) - F(a)$. Hence the result.

3. $\int_{-a}^{+a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(x)$ is an even function of x.

If $f(x)$ is even, $f(x) = f(-x)$.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \text{ by (2)}$$

$$= \int_{-a}^{0} f(-x)\,dx + \int_0^a f(x)\,dx$$

$$= -\int_a^0 f(y)\,dy + \int_0^a f(x)\,dx \quad \text{(by putting } y = -x \text{ in the first integral)}$$

$$= \int_0^a f(y)\,dy + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

as in a definite integral we can replace the variable y by x.

4. If $f(x)$ is an odd function of x, $\int_{-a}^{a} f(x)\,dx = 0$.

If $f(x)$ is odd, $f(x) = -f(-x)$.

$$\therefore \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

$$= -\int_{-a}^{0} f(-x)\,dx + \int_0^a f(x)\,dx$$

$$= \int_a^0 f(y)\,dy + \int_0^a f(x)\,dx \quad \text{(on putting } y = -x \text{ in the first integral)}$$

$$= 0.$$

5. $\int_0^a \mathbf{f}(\mathbf{x})\,\mathbf{dx} = \int_0^a \mathbf{f}(\mathbf{a} - \mathbf{x})\,\mathbf{dx}$.

In $\int_0^a f(a-x)\,dx$, put $a - x = y$.

$$\text{R.H.S.} = -\int_a^0 f(y)\,dy = \int_0^a f(y)\,dy = \int_0^a f(x)\,dx.$$

This result is very useful in evaluating many integrals.

Examples.

Ex. 1. Prove that $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx.$

Let $f(x) = \sin^n x$. Here $a = \frac{\pi}{2}$.

$$\therefore\ f(a-x) = \sin^n\left(\frac{\pi}{2} - x\right) = \cos^n x.$$

By § 52·5, the result follows.

Ex. 2. $\displaystyle\int_0^{\pi/2} \frac{(\sin x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}}\,dx = \frac{\pi}{4}.$ (B.Sc. Sub. 47)

Let I be the value of this integral and $f(x)$ denote the integrand $\dfrac{(\sin x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}}$.

$$\therefore\ I = \int_0^{\pi/2} f(x)\,dx. \qquad (1)$$

$$f(a-x) = \frac{(\cos x)^{3/2}}{(\cos x)^{3/2} + (\sin x)^{3/2}} \text{ as } a = \frac{\pi}{2} \text{ here.}$$

$$\text{Also } I = \int_0^{\pi/2} f(a-x)\,dx. \qquad (2)$$

Adding (1) and (2),

$$2I = \int_0^{\pi/2} \frac{(\sin x)^{3/2} + (\cos x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}}\,dx = \int_0^{\pi/2} dx = \Big(x\Big)_0^{\pi/2} = \frac{\pi}{2}.$$

Hence $I = \pi/4$.

Ex. 3. $\displaystyle\int_0^{\pi/4} \log(1 + \tan\theta)\,d\theta = \frac{\pi}{8}\log 2.$ (B.Sc. 53 30)

Let $f(\theta) = \log(1 + \tan\theta)$. Here $a = \frac{\pi}{4}$.

$$\therefore\ f\left(\frac{\pi}{4} - \theta\right) = \log\left\{1 + \tan\left(\frac{\pi}{4} - \theta\right)\right\}$$

$$= \log \left\{ 1 + \frac{\tan \frac{\pi}{4} - \tan \theta}{1 + \tan \frac{\pi}{4} \tan \theta} \right\} = \log \frac{2}{1 + \tan \theta}.$$

$$I = \int_0^{\pi/4} \log (1 + \tan \theta)\, d\theta$$

and $I = \int_0^{\pi/4} \log \frac{2}{1 + \tan \theta}\, d\theta$ by § 52·5.

Adding, $2I = \int_0^{\pi/4} \log 2\, d\theta = \log 2\, (\theta)_0^{\pi/4}$

$$= \frac{\pi}{4} \log 2.$$

Hence the result.

Ex. 4. $\int_0^{\pi} \theta \sin^3 \theta\, d\theta = \frac{2\pi}{3}$.

$f(\theta) = \theta \sin^3 \theta$. Here $a = \pi$.

$\therefore\ f(a - \theta) = (\pi - \theta) \sin^3 \theta$.

Hence $I = \int_0^{\pi} \theta \sin^3 \theta\, d\theta$ and $I = \int_0^{\pi} (\pi - \theta) \sin^3 \theta\, d\theta$

by § 52·5.

Adding, $2I = \pi \int_0^{\pi} \sin^3 \theta\, d\theta$

$= \pi \int \sin^2 \theta\, (-dy)$ putting $\cos \theta = y$;

$-\sin \theta\, d\theta = dy$

$$= -\pi \int_1^{-1} (1 - y^2)\, dy = -\pi \left[y - \frac{y^3}{3} \right]_1^{-1}$$

$$= -\pi \left[-1 + \frac{1}{3} - 1 + \frac{1}{3} \right] = \frac{4\pi}{3}.$$

Hence $I = 2\pi/3$.

Ex. 5. $\int_0^{2a} \mathbf{f}(\mathbf{x})\, \mathbf{dx} = 2 \int_0^{a} \mathbf{f}(\mathbf{x})\, \mathbf{dx}$ **if** $\mathbf{f}(2\mathbf{a} - \mathbf{x}) = \mathbf{f}(\mathbf{x})$

and $= 0$ **if** $\mathbf{f}(2\mathbf{a} - \mathbf{x}) = -\mathbf{f}(\mathbf{x})$.

$$\int_0^{2a} f(x)\, dx = \int_0^{a} f(x)\, dx + \int_a^{2a} f(x)\, dx.$$

In the second integral, put $2a - x = y$; $dx = -dy$.
When $x = a$, $y = a$ and $x = 2a$, $y = 0$.

Hence $\int_a^{2a} f(x)\,dx = -\int_a^0 f(2a - y)\,dy = \int_0^a f(2a - y)\,dy$

$$= \int_0^a f(2a - x)\,dx.$$

$\therefore \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a - x)\,dx$ from (1).

If $f(2a - x) = f(x)$, $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

If $f(2a - x) = -f(x)$, $\int_0^{2a} f(x)\,dx = 0$.

Cor. $\int_0^{\pi} f(\sin x)\,dx = 2\int_0^{\pi/2} f(\sin x)\,dx$.

Ex. 6. Evaluate $I = \int_0^{\pi/2} \log \sin x\,dx$. (B.Sc. 52 M)

$I = \int_0^{\pi/2} \log \sin(\pi/2 - x)\,dx = \int_0^{\pi/2} \log \cos x\,dx$ (by 5).

Hence $2I = \int_0^{\pi/2} \log \sin x\,dx + \int_0^{\pi/2} \log \cos x\,dx$.

$$= \int_0^{\pi/2} \log(\sin x \cos x)\,dx$$

$$= \int_0^{\pi/2} (\log \sin 2x - \log 2)\,dx$$

$$= \int_0^{\pi/2} \log \sin 2x\,dx - \frac{\pi}{2}\log 2.$$

Put $2x = z$; $dx = \frac{1}{2}dz$; then

$\int_0^{\pi/2} \log \sin 2x\,dx = \frac{1}{2}\int_0^{\pi} \log \sin z\,dz = \frac{1}{2}\int_0^{\pi} \log \sin x\,dx$

$$= \frac{1}{2} \times 2\int_0^{\pi/2} \log \sin x\,dx \qquad \text{(by 6 Cor.)}$$

$$= \int_0^{\pi/2} \log \sin x\,dx.$$

Thus, $2I = I - \frac{\pi}{2}\log 2$

i.e., $I = -\frac{\pi}{2}\log 2 = \frac{\pi}{2}\log(\frac{1}{2})$.

Exercises LXI.

(1) Prove that $\int_0^{\pi/2} \cos^2 x\, dx = \int_0^{\pi/2} \sin^2 x\, dx = \frac{\pi}{4}$.

(B.Sc. Sub. 44)

(2) Evaluate (i) $\int_0^{\pi/2} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\, dx$. (ii) $\int_0^{\pi/2} \frac{a \sin x + b \cos x}{\sin x + \cos x}\, dx$.

(3) $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$.

(B.Sc. 56 M)

(4) $\int_0^{a} x (a - x)^n\, dx$.

(5) $\int_0^{2} x \sqrt{2 - x}\, dx$.

(6) $\int_0^{1} x (1 - x)^n\, dx$.

(7) $\int_0^{\pi} \frac{x}{1 + \sin x}\, dx$.

(B.Sc. Sub. 51)

(8) $\int_0^{\pi} \frac{x \tan x}{\sec x + \cos x}\, dx$.

(B.Sc. 51 M)

(9) $\int_0^{\pi} \frac{x \sin^3 x}{1 + \cos^2 x}\, dx$.

(10) $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$.

(11) $\int_0^{1} \frac{\log(1 + x)}{1 + x^2}\, dx$.

(B.A. 52 M)

[*Hint*. Put $x = \tan \theta$ and use Ex. 3.]

(12) $\int_0^{\pi/2} \log \tan x\, dx$.

(13) $\int_0^{\infty} \log\left(x + \frac{1}{x}\right) \frac{dx}{1 + x^2}$.

[*Hint*. Put $x = \tan \theta$.]

(14) $\int_0^{\pi/2} \sin 2x \log \tan x\, dx$.

(15) Show that $\int_0^{1} x^m (1 - x)^n\, dx = \int_0^{1} x^n (1 - x)^m\, dx$;

hence find $\int_0^{1} x (1 - x)^4\, dx$.

(16) $\displaystyle\int_0^{\pi} \frac{x\,dx}{a^2 - \cos^2 x}$.

Prove the following results :—

(17) $\displaystyle\int_0^{\pi} \log(1 + \cos x)\,dx = \pi \log(1/2)$.

(18) $\displaystyle\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\,dx = \frac{1}{\sqrt{2}} \log(\sqrt{2} + 1)$.

(19) $\displaystyle\int_0^{\pi/2} \frac{x\,dx}{\sin x + \cos x} = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2} + 1)$.

(20) $\displaystyle\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\,dx = \pi\left(\frac{\pi}{2} - 1\right)$.

(21) Show that $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ and hence evaluate $\displaystyle\int_{\pi/n}^{(n+1)\pi/n} x \sin^3 x\,dx$. (B.A. 55)

§ 53. Integration by parts.

If u and v are functions of x,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \text{ by the product rule.}$$

Integrating both sides with respect to x,

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx.$$

$\therefore \quad uv = \int u\,dv + \int v\,du.$

Hence $\int u\,dv = uv - \int v\,du.$

Note.—The success of this method depends on the proper choice of u and v; the auxiliary integral $\int v\,du$ must be easier to integrate than the given integral.

Examples.

Ex. 1. $\int xe^x\,dx$.

Writing $dv = e^x\,dx$ and $u = x$, $v = \int e^x\,dx = e^x$.

$$\therefore \int xe^x\,dx = \int x\,d(e^x) = \int u\,dv = uv - \int v\,du$$
$$= xe^x - \int e^x\,dx = xe^x - e^x = e^x(x-1).$$

Ex. 2. $\int x \sin 2x \, dx$.

Here $dv = \sin 2x \, dx$, *i.e.*, $v = \int \sin 2x \, dx = \dfrac{-\cos 2x}{2}$ and $u = x$.

$$\therefore \int x \sin 2x \, dx = \int x d\left(\frac{-\cos 2x}{2}\right) = \int u \, dv = uv - \int v \, du$$

$$= -\frac{x \cos 2x}{2} + \tfrac{1}{2}\int \cos 2x \, dx = -\frac{x \cos 2x}{2} + \frac{\sin 2x}{4}.$$

Ex. 3. $\int x^n \log x \, dx$. Put $u = \log x$ and $dv = x^n \, dx$ *i.e.*, $v = \int x^n \, dx = x^{n+1}/(n+1)$.

$$\therefore \int x^n \log x \, dx = \int \log x \, d\left(\frac{x^{n+1}}{n+1}\right)$$

$$= \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\int x^{n+1}\frac{1}{x}\,dx$$

$$= \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\int x^n \, dx$$

$$= \frac{x^{n+1}}{n+1}\log x - \frac{x^{n+1}}{(n+1)^2}.$$

Ex. 4. $\int \sin^{-1} x \, dx$.

Put $u = \sin^{-1} x$ and $dv = dx$, *i.e.*, $v = x$.

$$\int \sin^{-1} x \, dx = \int u \, dv = uv - \int v \, du = x \sin^{-1} x - \int \frac{x \, dx}{\sqrt{1-x^2}}$$

$$= x \sin^{-1} x - \int \sin \theta \, d\theta \text{ on putting } x = \sin \theta;$$

$$= x \sin^{-1} x + \cos \theta = x \sin^{-1} x + \sqrt{1-x^2}.$$

Ex. 5. $\int \tan^{-1} x \, dx$. [Here $u = \tan^{-1} x$ and $v = x$.]

$$= x \tan^{-1} x - \int \frac{x \, dx}{1+x^2} = x \tan^{-1} x - \tfrac{1}{2}\log(1+x^2).$$

Ex. 6. $\int x^2 \tan^{-1} x \, dx$. [Here $u = \tan^{-1} x$; $dv = x^2 \, dx$. $\therefore v = x^3/3$.]

$$= \int \tan^{-1} x d\left(\frac{x^3}{3}\right) = \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int\left(x - \frac{x}{1+x^2}\right)dx$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\left\{\frac{x^2}{2} - \frac{1}{2}\log(1+x^2)\right\}.$$

(B.Sc. 49 M)

Ex. 7. $\int (\log x)^2 \, dx$.

Here $u = (\log x)^2$ and $v = x$.

$$\therefore \quad \int (\log x)^2\, dx = x(\log x)^2 - \int x \,.\, 2\log x \,.\, \frac{1}{x}\, dx$$
$$= x(\log x)^2 - 2\int \log x\, dx$$
$$= x(\log x)^2 - 2\left(x\log x - \int x \,.\, \frac{1}{x}\, dx\right)$$
$$= x(\log x)^2 - 2x\log x + 2x.$$

Ex. 8. $\int \sqrt{a^2+x^2}\, dx.$

(*Vide* Ex. 2. Page 263 for another method.)

[Here $u = \sqrt{a^2+x^2}$ and $v = x$]

$$\text{Integral} = x\sqrt{a^2+x^2} - \int \frac{x^2\, dx}{\sqrt{a^2+x^2}}$$
$$= x\sqrt{a^2+x^2} - \int \frac{a^2+x^2-a^2}{\sqrt{a^2+x^2}}\, dx$$
$$= x\sqrt{a^2+x^2} - \int \sqrt{a^2+x^2}\, dx + a^2\int \frac{dx}{\sqrt{a^2+x^2}}$$
$$= x\sqrt{a^2+x^2} - \int \sqrt{a^2+x^2}\, dx + a^2 \sinh^{-1} x/a.$$

Transposing, we get

$$\int \sqrt{a^2+x^2}\, dx = \frac{1}{2}\left(x\sqrt{a^2+x^2} + a^2 \sin^{-1}\frac{x}{a}\right).$$

Ex. 9. $\int \frac{x+\sin x}{1+\cos x}\, dx.$ (B.Sc. 60 M)

$$I = \int \frac{x\, dx}{1+\cos x} + \int \frac{\sin x\, dx}{1+\cos x}$$
$$= \int \frac{x\, dx}{2\cos^2 \frac{x}{2}} + \int \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\, dx$$
$$= \int x d\left(\tan\frac{x}{2}\right) + \int \tan\frac{x}{2}\, dx$$
$$= x\tan\frac{x}{2} - \int \tan\frac{x}{2}\, dx + \int \tan\frac{x}{2}\, dx = x\tan\frac{x}{2}.$$

Ex. 10. $\int e^x \frac{x+1}{(x+2)^2}\, dx.$ (B.Sc. 49 M)

$$= \int e^x \frac{x+2-1}{(x+2)^2}\, dx = \int \frac{e^x}{x+2}\, dx - \int \frac{e^x}{(x+2)^2}\, dx$$
$$= \int \frac{1}{x+2}\, d(e^x) - \int \frac{e^x}{(x+2)^2}\, dx$$

$$=\frac{e^x}{x+2}+\int\frac{e^x}{(x+2)^2}dx-\int\frac{e^x\,dx}{(x+2)^2}=\frac{e^x}{x+2}.$$

Ex. 11. $\int e^x(\sin x+\cos x)\,dx=\int e^x\sin x\,dx+\int e^x\cos x\,dx$
$=\int\sin x\,d(e^x)+\int e^x\cos x\,dx$
$=\sin x\,e^x-\int e^x\cos x\,dx+\int e^x\cos x\,dx=\sin x\,e^x.$

Exercises LXII. Integrate

1. $\log x.$
2. $x^3\log x.$
3. $x\log(x+1).$
4. $\cos^{-1}\frac{x}{a}.$
5. $x\sin^{-1}x.$
6. $x\tan^{-1}x.$
7. $x^2\sin^{-1}x.$
8. $x\sec x\tan x.$
9. $x\sin^2 x.$
10. $\frac{xe^x}{(1+x)^2}.$
11. $\sqrt{a^2-x^2}.$ (*Vide* Page 262)
12. $\frac{x^4\tan^{-1}x}{1+x^2}.$
13. $\frac{e^x(1+\sin x)}{1+\cos x}.$ (B.Sc. 40 M)
14. Evaluate $\int_0^\infty e^{-x^2}x^3\,dx.$ (B.Sc. Sub. 38)
15. $\frac{x\sin^{-1}x}{\sqrt{1-x^2}}.$ (B.Sc. Sub. 39)
16. Evaluate $\int_0^{\pi/4}\theta\sec^2\theta\,d\theta.$
17. Evaluate $\int_{1/2}^{1}\sin^{-1}\sqrt{x}\,dx.$ (B.A. 54 M)
18. $e^x(x+1)\log x.$
19. $\frac{x\tan^{-1}x}{(1+x^2)^{3/2}}.$
20. $\frac{x^2\tan^{-1}x}{1+x^2}.$
21. $\frac{x^3\sin^{-1}x}{\sqrt{1-x^2}}.$
22. $\frac{\log x}{x^2}.$
23. $\frac{\log x}{(x+1)^2}.$ (B.E. 48)
24. $x^n(\log x)^2.$
25. $\frac{\log(x^2+a^2)}{x^2}.$
26. $x\tan^2 x.$
27. $\frac{\tan^{-1}x}{x^2}.$
28. $\frac{x^3\sin^{-1}x}{(1-x^2)^{3/2}}.$ (B.Sc. 53 M)
29. $(\sin^{-1}x)^2.$
30. $x^2\sec^{-1}x.$

§ 54. Reduction formulae.

§ 54·1. $I_n=\int x^n e^{ax}\,dx$, where n is a positive integer.

Here $dv=e^{ax}\,dx$, i.e., $v=\int e^{ax}/a\,dx=e^{ax}/a$ and $u=x^n$.

$$\therefore\ I_n=\int x^n d\left(\frac{e^{ax}}{a}\right)=\frac{e^{ax}}{a}x^n-\frac{n}{a}\int e^{ax}x^{n-1}\,dx$$

$$=\frac{e^{ax}}{a}x^n-\frac{n}{a}I_{n-1}.$$

The auxiliary integral is of the same type as the given integral but with index n reduced by 1. Such a formula is called a *reduction formula* and by successive applications, we can evaluate I_n. The ultimate integral is obviously

$$\int e^{ax}\,dx = e^{ax}/a.$$

§ 54·2. $I_n = \int x^n \cos ax\, dx$ (n a positive integer).

$$I_n = \int x^n \cos ax\, dx = \int x^n\, d(\sin ax/a)$$

[Here $u = x^n$ and $v = \sin ax/a$.]

$$= \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1} \sin ax\, dx$$

$$= \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1}\, d\left(\frac{-\cos ax}{a}\right)$$

$$= \frac{x^n \sin ax}{a} + \frac{n}{a^2} x^{n-1} \cos ax - \frac{n(n-1)}{a^2}\int x^{n-2} \cos ax\, dx$$

$$= \frac{x^n \sin ax}{a} + \frac{n}{a^2} x^{n-1} \cos ax - \frac{n(n-1)}{a^2} I_{n-2}.$$

The ultimate integral is either $\int x \cos ax\, dx$ or $\int \cos ax\, dx$ according as n is odd or even.

(*i*) $\displaystyle\int x \cos ax\, dx = \int x d\left(\frac{\sin ax}{a}\right) = \frac{x \sin ax}{a} - \frac{1}{a}\int \sin ax\, dx$

$$= \frac{x \sin ax}{a} + \frac{1}{a^2} \cos ax.$$

(*ii*) $\displaystyle\int \cos ax\, dx = \frac{\sin ax}{a}.$

Exercises LXIII. Integrate

1. $x^2 e^{-x}$.
2. $x^3 e^{2x}$.
3. $e^x (x-1)^2$.
4. $x \cos 2x$.
5. $x^2 \sin 3x$.
6. $x^3 \cos (x+a)$.
7. Establish a reduction formula for $\int x^n \sin ax\, dx$.
8. If $I_n = \int_0^{\pi/2} x^n \cos x\, dx$, show that $I_n + n(n-1) I_{n-2}$ $= (\pi/2)^n$. Evaluate $\int_0^{\pi/4} x^3 \cos^2 x\, dx$. (B.E. 49)
9. Evaluate $\int_0^1 x^n e^x\, dx$.

§ 54·3. $I_n = \int \sin^n x\, dx$ (*n* being a positive integer).

$$I_n = \int \sin^{n-1} x \sin x\, dx = \int \sin^{n-1} x\, d(-\cos x)$$
$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x\, dx$$
$$= -\sin^{n-1} x \cos x + (n-1) \int (1-\sin^2 x) \sin^{n-2} x\, dx$$
$$= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n.$$

$$\therefore\ \mathbf{n I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}}.$$

The ultimate integral is $\int \sin x\, dx$ or $\int dx$ according as *n* is odd or even, *i.e.*, $-\cos x$ or x.

Corollary.

$$\int_0^{\pi/2} \sin^n x\, dx = \left[-\frac{\sin^{n-1} x \cos x}{n}\right]_0^{\pi/2} + \frac{(n-1)}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$= \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx \text{ as the first term vanishes at both limits}$$

$$= \frac{n-1}{n}\frac{n-3}{n-2} \int_0^{\pi/2} \sin^{n-4} x\, dx$$

$$= \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4} \cdots\cdots$$

If *n* is even, the ultimate integral is

$$\int_0^{\pi/2} dx = \left[x\right]_0^{\pi/2} = \frac{\pi}{2}.$$

If *n* is odd, the ultimate integral is

$$\int_0^{\pi/2} \sin x\, dx = \left(-\cos x\right)_0^{\pi/2} = 1.$$

$$\therefore \int_0^{\pi/2} \mathbf{\sin^n x\, dx} = \frac{\mathbf{n-1}}{\mathbf{n}}\frac{\mathbf{n-3}}{\mathbf{n-2}} \cdots \frac{1}{2}\frac{\pi}{2}, \text{ when } n \text{ is even and}$$

$$\frac{\mathbf{n-1}}{\mathbf{n}}\frac{\mathbf{n-3}}{\mathbf{n-2}} \cdots \frac{2}{3}, \text{ when } n \text{ is odd.}$$

Examples.

Ex. 1. $\displaystyle\int_0^{\pi/2} \sin^6 x\, dx = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{32}.$

Ex. 2. $$\int_0^{\pi/2} \sin^7 x\, dx = \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3} = \frac{48}{105}.$$

Ex. 3. In $\int \sin^n x\, dx$, if n be an odd positive integer, we can directly integrate without using the reduction formula. For instance, let us find $\int \sin^5 x\, dx$.

Put $y = \cos x$; $dy = -\sin x\, dx$.

$$\int \sin^5 x\, dx = -\int \sin^4 x\, dy = -\int (1-y^2)^2\, dy$$
$$= -\int (1 - 2y^2 + y^4)\, dy$$
$$= -y + \frac{2y^3}{3} - \frac{y^5}{5} = -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5}.$$

Ex. 4. Evaluate $\int_0^1 x\,(1-x^2)^{1/2}\, dx$. (B.Sc. Sub. 50)

Put $x = \sin\theta$; $dx = \cos\theta\, d\theta$.

When $x = 0$, $\theta = 0$ and $x = 1$, $\theta = \pi/2$.

The integral becomes

$$\int_0^{\pi/2} \sin\theta\cos^2\theta\, d\theta = \int \cos^2\theta\, d(-\cos\theta) = \left[\frac{-\cos^3\theta}{3}\right]_0^{\pi/2} = \frac{1}{3}.$$

§ 54·4. $\mathbf{I_n = \int \cos^n x\, dx}$ (*n* being a positive integer).

$$I_n = \int \cos^n x\, dx = \int \cos^{n-1} x \cos x\, dx$$
$$= \int \cos^{n-1} x\, d(\sin x)$$
$$= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\, dx$$
$$= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\, dx$$
$$= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$\therefore\ \mathbf{nI_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}}.$$

The ultimate integral is $\int \cos x\, dx$ or $\int dx$, *i.e.*, $\sin x$ or x according as n is odd or even.

Corollary.

$$\int_0^{\pi/2} \cos^n x\, dx = \left(\frac{\cos^{n-1} x \sin x}{n}\right)_0^{\pi/2} + \frac{(n-1)}{n}\int_0^{\pi/2} \cos^{n-2} x\, dx$$

$$= \frac{(n-1)}{n}\int_0^{\pi/2} \cos^{n-2} x\, dx$$ as the first term vanishes at both limits.

$$=\frac{n-1}{n}\frac{n-3}{n-2}\int_0^{\pi/2}\cos^{n-4}x\,dx$$

$$=\frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots$$

The ultimate integral is

$$\int_0^{\pi/2}\cos x\,dx=\Big[\sin x\Big]_0^{\pi/2}=1 \text{ when } n \text{ is odd.}$$

The ultimate integral is

$$\int_0^{\pi/2}dx=\Big[x\Big]_0^{\pi/2}=\frac{\pi}{2}, \text{ when } n \text{ is even.}$$

Thus $\displaystyle\int_0^{\pi/2}\cos^n x\,dx=\frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{1}{2}\frac{\pi}{2}$, if n is even

and $\displaystyle =\frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{2}{3}$, if n is odd.

Examples.

Ex. 1. $\displaystyle\int_0^{\pi/2}\cos^8 x\,dx=\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{35\pi}{256}.$

Ex. 2. $\displaystyle\int_0^{\pi/2}\cos^5 x\,dx=\frac{4}{5}\cdot\frac{2}{3}=\frac{8}{15}.$

Ex. 3. In $\int\cos^n x\,dx$, if n be an odd positive integer, we can directly integrate without employing the reduction formula. For example, take $\int\cos^7 x\,dx$.

Put $y=\sin x$; $dy=\cos x\,dx$.

$$\int\cos^7 x\,dx=\int\cos^6 x\cos x\,dx=\int(1-y^2)^3\,dy$$

$$=\int(1-3y^2+3y^4-y^6)\,dy=y-y^3+\frac{3y^5}{5}-\frac{y^7}{7}$$

$$=\sin x-\sin^3 x+\frac{3\sin^5 x}{5}-\frac{\sin^7 x}{7}.$$

§ 54·5. $I_{m,n}=\int\sin^m x\cos^n x\,dx$ (m, n being positive integers).

$$I_{m,n} = \int \sin^m x \cos^{n-1} x \, d(\sin x)$$

$$= \int \cos^{n-1} x \, d\left(\frac{\sin^{m+1} x}{m+1}\right)$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} - \frac{1}{m+1} \int \sin^{m+1} x \, d(\cos^{n-1} x)$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{m+2} x \cos^{n-2} x \, dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x (1 - \cos^2 x) \, dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x \, dx - \frac{n-1}{m+1} \int \sin^m x \cos^n x \, dx.$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}.$$

$$\therefore \quad (\mathbf{m} + \mathbf{n}) \, \mathbf{I}_{m,n} = \mathbf{\cos^{n-1} x \sin^{m+1} x + (n-1) \, I_{m,n-2}}. \quad (a)$$

Here, the power of $\cos x$ has been reduced by 2. We may, by a similar argument, arrive at the reduction formula in the form

$$(\mathbf{m} + \mathbf{n}) \, \mathbf{I}_{m,n} = \mathbf{-\sin^{m-1} x \cos^{n+1} x + (m-1) \, I_{m-2,n}}. \quad (b)$$

Here, the power of $\sin x$ has been reduced by 2.

To apply this formula, we note two cases.

Cass (i). *Let m or n be an odd integer, say, n.*

Applying the formula (a) successively, the ultimate integral is

$$I_{m,1} = \int \sin^m x \cos x \, dx = \frac{\sin^{m+1} x}{m+1}.$$

If, however, m is odd, we can use (b) and the ultimate integral is

$$I_{1,n} = \int \sin x \cos^n x \, dx = -\frac{\cos^{n+1} x}{n+1}.$$

If both m and n are odd, reduce the smaller index.

Note.—When either m or n or both are odd, we can integrate $\sin^m x \cos^n x$ directly without recourse to a reduction formula. For example, take

(1) $\int \sin^6 x \cos^3 x \, dx$. Put $y = \sin x$; $dy = \cos x \, dx$.

$$\int \sin^6 x \cos^3 x \, dx = \int y^6 (1 - y^2) \, dy = \frac{y^7}{7} - \frac{y^9}{9}$$

$$= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9}.$$

(2) $\int \sin^9 x \cos^5 x\, dx$. Put $\sin x = y$; $\cos x\, dx = dy$.

$$\int \sin^9 x \cos^5 x\, dx = \int y^9 (1-y^2)^2\, dy = \int y^9 (1 - 2y^2 + y^4)\, dy$$

$$= \frac{y^{10}}{10} - \frac{y^{12}}{6} + \frac{y^{14}}{14} = \frac{\sin^{10} x}{10} - \frac{\sin^{12} x}{6} + \frac{\sin^{14} x}{14}.$$

Case (ii). *Let both m and n be even $+ve$ integers.*

Let $n < m$. Applying (a), the ultimate integral is

$I_{m,0} = \int \sin^m x\, dx$ which has been discussed in § 54·3.

Corollary.

$\int_0^{\pi/2}$ **$\sin^m x \cos^n x\, dx$** (m, n being positive integers).

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx$$

$$= \left[\frac{\cos^{n-1} x \sin^{m+1} x}{m+n}\right]_0^{\pi/2} + \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x \cos^{n-2} x\, dx$$

$$= \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x \cos^{n-2} x\, dx \text{ as the first term vanishes at both limits}$$

$$= \frac{n-1}{m+n}\cdot\frac{n-3}{m+n-2}\int_0^{\pi/2} \sin^m x \cos^{n-4} x\, dx$$

$$= \frac{n-1}{m+n}\,\frac{n-3}{m+n-2}\,\frac{n-5}{m+n-4}\ldots\ldots I_{m,1} \text{ or } I_{m,0} \text{ according as } n \text{ is odd or even.}$$

(i) If n is odd, $I_{m,1} = \int_0^{\pi/2} \sin^m x \cos x\, dx$.

$$= \left[\frac{\sin^{m+1} x}{m+1}\right]_0^{\pi/2} = \frac{1}{m+1}.$$

When n is odd,

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{n-1}{m+n}\,\frac{n-3}{m+n-2}\cdots\frac{2}{m+3}\,\frac{1}{m+1}.$$

(ii) If n is even,

$$I_{m,0} = \int_0^{\pi/2} \sin^m x\, dx = \frac{m-1}{m}\,\frac{m-3}{m-2}\cdots\frac{1}{2}\frac{\pi}{2} \text{ by § 54·3 Cor.}$$

When n is even,

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

$$= \frac{n-1}{m+n} \frac{n-3}{m+n-2} \cdots \frac{1}{m+2} \cdot \frac{m-1}{m} \cdot \frac{m-3}{m-2} \cdots \frac{1}{2}\frac{\pi}{2}.$$

Examples.

Ex. 1. $\int_0^{\pi/2} \sin^6 x \cos^5 x \, dx = \frac{4}{11} \cdot \frac{2}{9} \cdot \frac{1}{7}$ by (*i*)

$$= \frac{8}{693}.$$

Ex. 2. $\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx = \frac{5}{10} \cdot \frac{3}{8} \cdot \frac{1}{6} \cdot \frac{3}{4} \cdot \frac{1}{2}\frac{\pi}{2} = \frac{3\pi}{512}.$

§ 54·6. $I_n = \int \tan^n x \, dx$ (*n* being a positive integer).

$$I_n = \int \tan^{n-2} x \tan^2 x \, dx$$

$$= \int \tan^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \int \tan^{n-2} x \, d(\tan x) - \int \tan^{n-2} x \, dx$$

$$= \frac{\tan^{n-1} x}{n-1} - I_{n-2}.$$

(*i*) When n is even, the ultimate integral is $\int dx = x$.

(*ii*) When n is odd, the ultimate integral is

$$\int \tan x \, dx = \log \sec x. \quad (\textit{Vide}\ \S\ 47{\cdot}4,\ \text{Ex. } 3)$$

Examples.

Ex. 1. $\int \tan^4 x \, dx = \frac{\tan^3 x}{3} - \int \tan^2 x \, dx$ by putting $n = 4$ in the formula for I_n

$$= \frac{\tan^3 x}{3} - \int (\sec^2 x - 1) \, dx$$

$$= \frac{\tan^3 x}{3} - \tan x + x.$$

Ex. 2. $\int_0^{\pi/4} \tan^3 x \, dx = \left[\frac{\tan^2 x}{2}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan x \, dx$

by putting $n = 3$

$$= \frac{1}{2} + \left[\log \cos x\right]_0^{\pi/4} = \frac{1}{2} + \log \frac{1}{\sqrt{2}} = \frac{1}{2}(1 - \log 2).$$

§ 54·7. $\mathbf{I_n = \int \cot^n x\, dx}$ (*n* being a positive integer).

$$\int \cot^n x\, dx = \int \cot^{n-2} x \cot^2 x\, dx$$
$$= \int \cot^{n-2} x\,(\operatorname{cosec}^2 x - 1)\, dx$$
$$= \int \cot^{n-2} x\, d(-\cot x) - \int \cot^{n-2} x\, dx$$
$$= -\frac{\cot^{n-1} x}{n-1} - I_{n-2}.$$

The ultimate integral is $\int dx$ or $\int \cot x\, dx$, *i.e.*, x or $\log \sin x$ according as n is even or odd.

§ 54·8. $\mathbf{I_n = \int \sec^n x\, dx}$ (*n* being a positive integer).

$$\int \sec^n x\, dx = \int \sec^{n-2} x\, d(\tan x)$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\, dx$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x\,(\sec^2 x - 1)\, dx$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\, dx + (n-2)\int \sec^{n-2} x\, dx$$
$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}.$$

$$\therefore \quad \mathbf{(n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}}.$$

(*i*) If n be an odd integer, the ultimate integral is $\int \sec x\, dx = \log(\tan x + \sec x)$. (*Vide* § 47·4, Ex. 5)

(*ii*) If n be an even integer, the ultimate integral is $\int dx = x$.

Examples.

Ex. 1. $$I = \int \sec^3 x\, dx = \int \sec x\, d(\tan x)$$
$$= \sec x \tan x - \int \tan^2 x \sec x\, dx$$
$$= \sec x \tan x - \int (\sec^2 x - 1)\sec x\, dx$$
$$= \sec x \tan x - I + \log(\sec x + \tan x).$$
$$\therefore\ 2I = \sec x \tan x + \log(\sec x + \tan x).$$

Ex. 2. $$\int \sec^6 x\, dx = \int \sec^4 x\, d(\tan x) = \int (1+t^2)^2\, dt$$
(where $t = \tan x$) ;
$$= \int (1 + 2t^2 + t^4)\, dt = t + \frac{2t^3}{3} + \frac{t^5}{5}.$$

§ 54·9. $\mathbf{I_n = \int \operatorname{cosec}^n x\, dx}$ (*n* being a positive integer).

$$I_n = \int \operatorname{cosec}^n x\, dx = -\int \operatorname{cosec}^{n-2} x\, d(\cot x)$$
$$= -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int \operatorname{cosec}^{n-2} x \cot^2 x\, dx$$
$$= -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int \operatorname{cosec}^{n-2} x \times (\operatorname{cosec}^2 x - 1)\, dx$$
$$= -\operatorname{cosec}^{n-2} x \cot x - (n-2) I_n + (n-2) I_{n-2}.$$
$$\therefore \quad \mathbf{(n-1) I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_{n-2}}.$$

(*i*) If n be an odd integer, the ultimate integral is $\int \operatorname{cosec} x\, dx = -\log(\operatorname{cosec} x + \cot x)$. (*Vide* § 47·4, Ex. 6)

(*ii*) If n be an even integer, ultimate integral is $\int dx = x$.

Example.

Ex. 1. $\int \operatorname{cosec}^4 x\, dx = -\int \operatorname{cosec}^2 x\, d(\cot x)$

$= -\int (1 + y^2)\, dy$ where $y = \cot x$.

$$= -y - \frac{y^3}{3} = -\cot x - \frac{\cot^3 x}{3}.$$

Ex. 2. $\int \operatorname{cosec}^5 x\, dx$.

Putting $n = 5$ in the above formula for I_n,

$$\int \operatorname{cosec}^5 x\, dx = -\frac{\operatorname{cosec}^3 x \cot x}{4} + \frac{3}{4}\int \operatorname{cosec}^3 x\, dx$$

$$= -\frac{\operatorname{cosec}^3 x \cot x}{4} - \frac{3}{8}\operatorname{cosec} x \cot x$$

$$- \frac{3}{8}\log(\operatorname{cosec} x + \cot x).$$

§ 54·10. $I_{m,\,n} = \int x^m (\log x)^n\, dx$ (where m and n are positive integers).

Hence or otherwise evaluate $\int x^4 (\log x)^3\, dx$.

$$I_{m,\,n} = \int (\log x)^n\, d\left(\frac{x^{m+1}}{m+1}\right)$$

$$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1}\int x^m (\log x)^{n-1}\, dx$$

$$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} I_{m,\,n-1}.$$

The ultimate integral is $I_{m,\,0} = \int x^m\, dx = \dfrac{x^{m+1}}{m+1}$.

Example. $\int (\log x)^3 x^4\, dx = \int (\log x)^3\, d\left(\frac{x^5}{5}\right)$

$$= \frac{x^5}{5}(\log x)^3 - \frac{3}{5}\int (\log x)^2 x^4\, dx$$

$$= \quad ,, \quad - \frac{3}{5}\int (\log x)^2\, d\left(\frac{x^5}{5}\right)$$

$$= \quad ,, \quad - \frac{3}{25}x^5 (\log x)^2 + \frac{6}{25}\int x^4 (\log x)\, dx$$

$$=\frac{x^5}{5}(\log x)^3+\frac{6}{25}\left\{\frac{x^5}{5}\log x-\frac{x^5}{25}\right\}$$

$$=x^5\left\{\frac{1}{5}(\log x)^3-\frac{3}{25}(\log x)^2+\frac{6}{125}\log x-\frac{6}{625}\right\}.$$

Exercises LXIV.

1. If $u_n=\int_0^a x^n e^{-x}\,dx$, prove that

$$u_n-(n+a)\,u_{n-1}+a\,(n-1)\,u_{n-2}=0. \quad \text{(B.Sc. 50 M)}$$

2. If $I_n=\int\frac{dx}{(x^2+1)^n}$, show that

$2n\,I_{n+1}=(2n-1)\,I_n+\frac{x}{(x^2+1)^n}$. Hence find

$\int_0^1\frac{dx}{(x^2+1)^3}$. (B.Sc. 47 M) [*Hint.* Put $x=\tan\theta$.]

3. If $u_n=\int_0^{\pi/2} x^n \sin x\,dx$ and n is a positive integer, prove that $u_n+n\,(n-1)\,u_{n-2}=n\left(\frac{\pi}{2}\right)^{n-1}$. (B.Sc. Sub. 41)

4. Find a reduction formula for $\int\frac{dx}{(x^2+1)^n}$.

5. Evaluate (*a*) $\int\sin^5 x\,dx$. (*b*) $\int_0^{\pi/2}\sin^4 x\,dx$. (*c*) $\int\cos^6 x\,dx$. (*d*) $\int_0^{\pi/2}\cos^7 x\,dx$. (*e*) $\int\sin^7 x\cos^3 x\,dx$. (*f*) $\int\sin^4 x\cos^3 x\,dx$. (*g*) $\int\sin^4 x\cos^4 x\,dx$. (*h*) $\int_0^{\pi/2}\sin^8 x\cos^4 x\,dx$. (*i*) $\int_0^{\pi/2}\sin^7 x\cos^5 x\,dx$. (*j*) $\int\sin^3 x\cos^3 x\,dx$. (B.Sc. 45 M) (*k*) $\int_{-\pi/2}^{\pi/2}\sin^2\theta\cos^7\theta\,d\theta$. (*l*) $\int_0^{\pi/2}\sin^2\theta\,(\sin^3\theta+\cos^3\theta)\,d\theta$.

6. Evaluate (*a*) $\int\tan^6 x\,dx$. (*b*) $\int_0^{\pi/4}\tan^5 x\,dx$. (*c*) $\int\cot^3 x\,dx$. (*d*) $\int\cot^4 x\,dx$. (*e*) $\int\sec^4 x\,dx$. (*f*) $\int\sec^5 x\,dx$; $\int_0^{\pi/4}\sec^5 x\,dx$. (B.A. 53 M) (*g*) $\int\operatorname{cosec}^3 x\,dx$. (*h*) $\int\operatorname{cosec}^6 x\,dx$. (*i*) $\int\sec x\tan^2 x\,dx$. (B.A. 51 M)

7. Evaluate $\int (\log x)^2 x^3\, dx$ and $\int (\log x)^5 x^6\, dx$.

8. $\int_0^1 x^3 (1 - x^2)^{-1/2}\, dx$.

9. $\int_0^1 \frac{x^6\, dx}{\sqrt{1 - x^2}}$. (B.A. 51 M)

[Put $x = \sin\theta$.]

10. $\int_0^1 x^2 \sin^{-1} x\, dx$.

11. $\int_0^1 \frac{x^3 \sin^{-1} x}{(1 - x^2)^{1/2}}\, dx$.

12. $\int_0^1 x^{3/2} \sqrt{1 - x}\, dx = \frac{\pi}{16}$.

(B.A. 36 M)

13. $\int_0^\pi x \sin^6 x \cos^4 x\, dx$.

14. $\int_0^\pi \cos^4 x \sin^4 x\, dx$.

15. (a) $\int_0^\infty \frac{x^3\, dx}{(a^2 + x^2)^5}$.

(b) $\int_1^\infty \frac{dx}{(a^2+x^2)^3}$. (B.E. 55)

16. $\int_0^1 \frac{x^{2n}}{\sqrt{1 - x^2}}\, dx$.

17. $\int_0^a x \sqrt{a^2 - x^2}\, dx$.

18. If $\int_0^{\pi/2} \cos^m x \cos nx\, dx = f(m, n)$, prove that $f(m, n)$ $= \frac{m}{m+n} f(m-1, n-1)$. Hence prove that $f(n, n) = \frac{\pi}{2^{n+1}}$.

(B.A. 49 M)

Solution.

$$\int \cos^m x \cos nx\, dx = \int \cos^m x\, d\left(\frac{\sin nx}{n}\right)$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \sin x \sin nx\, dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \{ \cos (n-1) x - \cos nx \cos x \}\, dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \cos (n-1) x\, dx - \frac{m}{n} \int \cos^m x \cos nx\, dx.$$

Hence $f(m, n) = \int_0^{\pi/2} \cos^m x \cos nx\, dx$

$$=\frac{1}{m+n}\left\{\left(\cos^m x \sin nx\right)_0^{\pi/2}+ m\int_0^{\pi/2}\cos^{m-1} x \cos (n-1)\, x\, dx\right\}$$

$=\frac{1}{m+n} f(m+1, n-1)$ as the first term vanishes at both limits.

Putting $m=n$, $f(n, n)=\frac{1}{2} f(n-1, n-1)=\frac{1}{2^2} f(n-2, n-2)=\frac{1}{2^n} f(0, 0)$

by repeated application of the same formula

$$=\frac{1}{2^n}\int_0^{\pi/2} dx=\frac{\pi}{2^{n+1}}.$$

19. Show that, if

$$u_n=\int_0^{2a} x^n \sqrt{2ax-x^2}\, dx,\quad u_n=\frac{2n+1}{n+2}\, a\, u_{n-1}.$$

[*Hint.* Put $x=2a \sin^2 \theta$.]

Find $\int_0^{2a} x\sqrt{2ax-x^2}\, dx$. (B.E. 46)

20. If $\int_0^{\pi/2} \cos^m x \sin nx\, dx=f(m, n)$, prove that

$f(m, n)=\frac{1}{m+n}+\frac{m}{m+n} f(m-1, n-1)$. Hence deduce that

$f(m, m)=\frac{1}{2^{m+1}}\left[\frac{2}{1}+\frac{2^2}{2}+\frac{2^3}{3}+\dots+\frac{2^m}{m}\right]$. (B.Sc. 49 M)

21. If $I_n=\int_{-1}^{+1}(1-x^2)^n\, dx$, prove that $I_n=\frac{2n}{2n+1} I_{n-1}$.

22. If $I_{m, n}=\int x^m (1+x)^n\, dx$, prove that

$$I_{m, n}=x^{m+1}(1+x)^n-n\, I_{m+1, n-1}.$$

23. If $I_n=\int\frac{e^{ax}}{x^n}\, dx$, show that

$$I_n=-\frac{e^{ax}}{(n-1)\, x^{n-1}}+\frac{a}{n-1} I_{n-1}.$$

24. If $I_n = \int_0^a (a^2 - x^2)^n \, dx$, prove that, if $n > 0$,

$$I_n = \frac{2n\,a^2}{2n+1} I_{n-1}.$$

25. If $I_n = \int_0^1 x^p \,(1 - x^q)^n \, dx$, where p, q, n are positive, prove that $(p + qn + 1)\, I_n = qn\, I_{n-1}$. Evaluate I_4.
(B.Sc. 52 T.U.)

26. If $I_n = \int_0^1 x^2 (1 - x^3)^n \, dx$, prove that

$$I_n = \frac{n}{n+1} I_{n-1}.$$ Hence evaluate $\int_0^1 x^2 \,(1 - x^3)^7 \, dx$.

(B.E. 49)

27. If $I_n = \int_0^{\pi/2} \theta \sin^n \theta \, d\theta$ and $n > 1$, prove that

$$I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n^2}.$$ Deduce that $I_5 = \frac{154}{225}$.

28. If $I_n = \int_0^{\pi/2} x \cos^n x \, dx$, where $n > 1$, show that

$$I_n = -\frac{1}{n^2} + \frac{n-1}{n} I_{n-2}.$$ Hence find I_4.

29. If $I_n = \int \frac{t^n}{1 + t^2} \, dt$, show that

$$I_{n+2} = \frac{t^{n+1}}{n+1} - I_n.$$ Evaluate I_6.

30. If $I_n = \int_0^\infty e^{-x} \sin^n x \, dx$, show that

$$(1 + n^2)\, I_n = n\,(n-1)\, I_{n-2}.$$ Hence find $\int_0^\infty e^{-x} \sin^4 x \, dx$.

(B.Sc. 56 M)

31. Find a reduction formula for $\int e^{ax} \cos^n x \, dx$ and deduce $\int_0^\infty e^{-x} \cos^3 x \, dx$.
(B.Sc. 62 M)

§ 55. Type $\int e^{ax} \cos bx \, dx$, a and b are constants.

Let $C = \int e^{ax} \cos bx \, dx$ and $S = \int e^{ax} \sin bx \, dx$.

$$C + iS = \int e^{ax} (\cos bx + i \sin bx)\, dx$$
$$= \int e^{ax} e^{ibx} \, dx$$ (by Euler's formula $e^{i\theta} = \cos\theta + i \sin\theta$

$$= \int e^{x(a+ib)}\, dx = \frac{e^{x(a+ib)}}{a+ib}$$

$$= e^{ax} \frac{(a-ib)(\cos bx + i \sin bx)}{a^2+b^2}.$$

$$C = \text{Real Part of } e^{ax} \frac{(a-ib)(\cos bx + i\sin bx)}{a^2+b^2}$$

$$= e^{ax}\frac{(a\cos bx + b\sin bx)}{a^2+b^2}.$$

$$S = \text{Imaginary Part of } e^{ax}\frac{(a-ib)(\cos bx + i\sin bx)}{a^2+b^2}$$

$$= e^{ax}\frac{a\sin bx - b\cos bx}{a^2+b^2}.$$

Examples.

Ex. 1. $\int e^{2x} \cos 3x\, dx.$

$$C = \int e^{2x}\cos 3x\, dx = \text{Real Part of } \int e^{2x} e^{3ix}\, dx$$

$$= \quad ,, \quad \int e^{x(2+3i)}\, dx$$

$$= \quad ,, \quad \frac{e^{x(2+3i)}}{2+3i}$$

$$= \quad ,, \quad \frac{e^{2x}}{13}(2-3i)(\cos 3x + i\sin 3x)$$

$$= \frac{e^{2x}}{13}(2\cos 3x + 3\sin 3x).$$

Ex. 2. $$\int e^{-x}\sin^2 x\, dx = \int e^{-x}\frac{(1-\cos 2x)}{2}\, dx$$

$$= -\frac{e^{-x}}{2} - \frac{1}{2}\int e^{-x}\cos 2x\, dx$$

$$= -\frac{e^{-x}}{2} - \frac{1}{2}e^{-x}\frac{-\cos 2x + 2\sin 2x}{5}$$ by putting $a = -1$ and $b = 2$ in the above formula.

Ex. 3. $$\int e^{ax}\cos mx\cos nx\, dx = \tfrac{1}{2}\int e^{ax}\{\cos(m+n)x + \cos(m-n)x\}\, dx.$$

$$= \frac{1}{2}e^{ax}\left\{\frac{a\cos(m+n)x + (m+n)\sin(m+n)x}{a^2+(m+n)^2} + \frac{a\cos(m-n)x + (m-n)\sin(m-n)x}{a^2+(m-n)^2}\right\}.$$

Exercises LXV. Integrate

1. $e^x \sin 2x$.
2. $e^{-3x} \sin \frac{x}{2}$.
3. $e^{4x} \cos 3x$.
4. $e^{m \cos^{-1} x}$. (B.A. 46 M)
5. $e^{ax} \sin (bx + c)$. (B.Sc. 53 M)
6. $e^x \cos^2 x$.
7. $e^{2x} \cos (3x + 4)$.
8. $e^x \sin 3x \cos 2x$.
9. $e^{2x} \cos 5x \cos 4x$.
10. $e^{-3x} \sin 3x \sin 2x$.
11. $a^x \sin x$. (B.Sc. 54 M)
12. $3^x \sin 2x$. (B.E. 55)

§ 56. Bernouilli's formula.

This formula is merely an extension of the formula of integration by parts.

Let dashes denote successive differentiation and suffixes denote successive integration;

e.g., $\frac{du}{dx} = u'$; $\frac{d^2u}{dx^2} = u''$ etc., and $\int v dx = v_1$;

$\iint v\,(dx)^2 = v_2$, etc.

$$\int u\, dv = uv - \int v\, du \text{ by § 53.}$$
$$= uv - \int u'\, d\,(v_1) \quad \text{with the above notation by § 53}$$
$$= uv - u'v_1 + \int v_1 du'$$
$$= uv - u'v_1 + \int u''\, d\,(v_2)$$
$$= uv - u'v_1 + u''v_2 - \int v_2 du'' \quad \text{by § 53}$$
$$= uv - v'v_1 + u''v_2 - u'''v_3 + \ldots\ldots$$

The advantage of the above formula can be seen in examples where the tediousness of successive integration by parts is avoided.

Examples.

Ex. 1. Evaluate $\int x^4 e^x\, dx$. Here $u = x^4$ and $v = e^x$. $u' = 4x^3$; $u'' = 12x^2$; $u''' = 24x$ and $u'''' = 24$ while $v_1 = v_2 = v_3 = v_4 = e^x$.

$\therefore \int x^4 e^x\, dx = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x$ by the above formula.

Ex. 2. Evaluate $\int x^3 \cos 2x\, dx$.

Here $u = x^3$ and $v = \frac{\sin 2x}{2}$.

$u' = 3x^2$; $u'' = 6x$; and $u''' = 6$ while

$v_1 = -\frac{\cos 2x}{4}$; $v_2 = -\frac{\sin 2x}{8}$ and $v_3 = \frac{\cos 2x}{16}$.

$$\therefore \int x^3 \cos 2x \, dx = \frac{x^3 \sin 2x}{2} - 3x^2\left(\frac{-\cos 2x}{4}\right) + 6x\left(\frac{-\sin 2x}{8}\right) - 6\,\frac{\cos 2x}{16}$$

$$= \frac{1}{2}\left[x^3 \sin 2x + \frac{3x^2 \cos 2x}{2} - \frac{3x \sin 2x}{2} - \frac{3 \cos 2x}{4}\right].$$

Exercises LXVI. Integrate

1. $x^3 e^{-2x}$. 2. $x^4 \sin x$. 3. $x^3 \sin 3x$. 4. $x^2 (e^x + e^{-x})$.
5. $x^5 \cos \frac{x}{2}$. 6. $x^3 \sin nx$. (B.Sc. 55 M)

Miscellaneous exercises on integration LXVII.

1. $\int \frac{\sin 4x}{\sin^6 x}\, dx$. (B.Sc. 53 M)

2. $\int_0^{\pi} f(\sin x)\, dx = 2\int_0^{\pi/2} f(\sin x)\, dx$. (B.Sc. 51 M)

3. The transformation $\sin x = y$ gives on substitution

$$\int_{-\pi}^{\pi} \cos^2 x \; dx = \int_0^0 \sqrt{1-y^2}\, dy = 0.$$

Explain this paradox. (B.Sc. 52 M)

4. $\int (\log x)^3\, dx$.

5. $\int \frac{x\,dx}{\sqrt{ax - x^2}}$.

6. $\int \sinh 2x \cos 3x\, dx$.

7. (*a*) $\int \cosh 2x \cos 2x\, dx$.
(*b*) $\int \cosh ax \sin bx\, dx$. (B.Sc. Anc. 61)

8. $\int \sec^{-1} x\, dx$.

9. $\int \sinh^{-1} x\, dx$.

10. $\int x \cos^{-1} x\, dx$.

11. $\int x \log (1 + x^2)\, dx$.

12. $\int x\sqrt{x^2 + 4x + 13}\, dx$.

13. $\int \frac{dx}{\sqrt{x^2 + 4x + 13}}$.

14. $\int \frac{dx}{(1 + \cos x)^2}$.

15. Show that

$$\int_0^a \frac{dx}{x + \sqrt{(a^2 - x^2)}} = \frac{\pi}{4}.$$

16. Obtain $\int \frac{x^3\,dx}{(x^2+1)^3}$ by the substitutions $x = \tan\theta$ and $u = x^2 + 1$ and verify that the results agree.

17. Show that the transformation $\frac{9-x}{4x-9} = z^2$ converts

$$\int_{9/5}^{36/13} \sqrt{\frac{9-x}{4x-9}}\, dx \text{ into } \int_0^1 \frac{-6z^2}{(z^2+1)(4z^2+1)}\, dz$$ and evaluate it.

18. $\int \dfrac{dx}{\sin^3 x \cos^5 x}$. [Put $u = \tan x$.]

19. $\int \dfrac{dx}{2 + 3\cos x + 4\sin x}$.

20. $\int \dfrac{dx}{x^4 + x^2 + 1}$. (B.Sc. 51 T.U.)

21. If m and n are integers, prove that $\int_0^{\pi} \cos mx \sin nx\, dx$ $= \dfrac{2n}{n^2 - m^2}$ or 0 according as $n - m$ is odd or even.

22. Prove that $\int_0^{\pi/2} \sin^{n+1}\theta\, d\theta < \int_0^{\pi/2} \sin^n \theta\, d\theta$ and hence show that $\frac{\pi}{2}$ lies between

$$\frac{2.2.4.4.6.6 \ldots 2n.2n}{3.3.5.5.7.7 \ldots (2n-1).(2n-1).(2n+1)}$$

and the fraction obtained from this by omitting the last factor in each of the numerator and denominator. (B.Sc. 51 T.U.) Deduce Walli's formula for π, *viz.*,

$$\sqrt{\frac{\pi}{2}} = \operatorname*{Lt}_{n\to\infty} \frac{2.4.6 \ldots 2n}{1.3.5 \ldots (2n-1)\sqrt{2n+1}}.$$

23. If $f(m, n) = \int x^m (1-x)^n\, dx$, show that

$$f(m, n) = \frac{x^{m+1}(1-x)}{m+n+1} + \frac{n}{m+n+1} f(m, n-1).$$

Deduce $\int_0^1 x^m (1-x)^n\, dx$. (B.E. 46)

24. $\int_{-\pi/2}^{\pi/2} x \sin x \cos x\, dx = \frac{\pi}{4}$.

25. Obtain a reduction formula for $I_m = \int_0^{\infty} e^{-x} \sin^m x\, dx$ where $m > 2$ in the form $(1 + m^2) I_m = m(m-1) I_{m-2}$ and hence find I_4.

26. Show that

$$\int_0^{\pi} \frac{dx}{(a + b\cos x)^2} = \frac{1}{(a^2 - b^2)^{3/2}} \int_0^{\pi} (a - b\cos\theta)\, d\theta,$$

where $\cos\theta = \dfrac{a\cos x + b}{a + b\cos x}$ and $b < a$. (B.E. 44)

27. $\int x \cot^{-1} x \, dx.$ (B.Sc. Anc. 59)

28. $\int \frac{x \, dx}{1+\sqrt{x}}.$ (B.Sc. Anc. 59)

29. $\int \frac{dx}{\sqrt{\sin^3 x \sin (x+c)}}.$ (B.A. Hons.)

30. $\int \frac{\sqrt{\tan x}}{\sin x \cos x} \, dx.$ (B.Sc. Comp. Math. 59)

31. $\int \frac{e^x (1-\sin x)}{1-\cos x} \, dx.$

32. $\int_{-\pi/4}^{\pi/4} \frac{dx}{\sin^4 x \cos^4 x}.$ (B.A. 55 M)

33. $\int e^x (x+1)^2 \, dx.$ (B.Sc. Sub. 55)

34. $\int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx.$ (B.Sc. 59 M)

35. $\int x^2 \sin^2 x \, dx.$ (B.Sc. Anc. 59)

36. Prove that $\int u \frac{d^2v}{dx^2} \, dx = u \frac{dv}{dx} - v \frac{du}{dx} + \int v \frac{d^2u}{dx^2} \, dx.$
(B.Sc. 56 M)

37. Evaluate $\int \frac{1+2\sin x}{2+3\sin x} \, dx.$ (B.Sc. 60)

38. $\int \frac{1+x}{1+\cos x} \, dx.$

39. $\int \frac{dx}{(x-4)^{3/2}(x+4)^{1/2}}.$ (B Sc. 66)

§ 57·1. Integration as summation.

In § 42·1, we have mentioned that integration may be considered either as the inverse of differentiation or as a process of summation. We now proceed to consider its second aspect.

Let $f(x)$ be a continuous function in the *closed* interval from $x=a$ to $x=b$ (*i.e.*, the end points are included). Hence the function is bounded in the interval. Let $b > a$. Divide the interval (a, b) into n sub-intervals $x_1 - a$, $x_2 - x_1$, $x_3 - x_2$, ..., $b - x_{n-1}$, where $a, x_1, x_2, \ldots, x_{n-1}, b$ are in ascending order of magnitude. Let ξ_r be any point in the sub-interval (x_{r-1}, x_r). Taking $a = x_0$ and $b = x_n$, consider the sum

$$f(\xi_1)(x_1 - a) + f(\xi_2)(x_2 - x_1) + \ldots + f(\xi_r)(x_r - x_{r-1}) + \ldots + f(\xi_n)(b - x_{n-1}).$$

This sum tends to a definite limit when the number n of the sub-intervals tends to infinity, *i.e.*, the length of each partial sub-interval tends to zero, as a and b are finite. This limit is called the *definite integral* of $f(x)$ with respect to x from $x=a$ to $x=b$ and is written as $\int_a^b f(x) \, dx$.

A definite integral is therefore defined as the limit of the sum of a series. Even in the case of simple functions, the evaluation of an integral from this definition is not quite easy. To simplify calculations, we can take all the sub-intervals to be of equal length and take ξ_r to coincide with x_{r-1}. If h denotes $(b-a)/n$, then this definition leads us to the result

$$\int_a^b f(x)\,dx = \underset{h\to 0}{\mathrm{Lt}}\; h \sum_{r=1}^{r=n} f(a+\overline{r-1}\,h).$$

Cor. Put $a=0$ and $b=1$.

$$\text{Then } \int_0^1 f(x)\,dx = \underset{n\to\infty}{\mathrm{Lt}}\; \frac{1}{n} \sum_{r=0}^{r=n-1} f\left(\frac{r}{n}\right).$$

Examples.

Ex. 1. Calculate $\int_a^b x\,dx$.

Dividing $b-a$ into n equal sub-intervals each of length h, $f(a+\overline{r-1}\,h) = a+(r-1)h$ as $f(x)=x$ here.

$$\int_a^b x\,dx = \underset{h\to 0}{\mathrm{Lt}}\; h\,[a+(a+h)+\ldots+(a+\overline{n-1}\,h)]$$

$$= \underset{h\to 0}{\mathrm{Lt}}\; h\,[na+h(1+2+\ldots+\overline{n-1})]$$

$$= \underset{h\to 0}{\mathrm{Lt}} \left[a(b-a)+\frac{(b-a)^2}{n^2}\,\frac{(n-1)n}{2}\right] \text{ as } h=\frac{b-a}{n}$$

$$= \underset{h\to 0}{\mathrm{Lt}} \left[a(b-a)+\frac{(b-a)^2}{2}\left(1-\frac{1}{n}\right)\right].$$

When $h\to 0$, $n\to\infty$ and $1/n\to 0$.

$$\text{Hence} \int_a^b x\,dx = a(b-a)+\frac{(b-a)^2}{2} = \frac{b^2-a^2}{2}.$$

Ex. 2. Evaluate $\int_0^{\pi/2} \sin x\,dx$.

Putting $nh=\pi/2$,

$$\int_0^{\pi/2} \sin x\,dx = \underset{h\to 0}{\mathrm{Lt}}\; h\,[0+\sin h+\sin 2h+\ldots+\sin(n-1)h]$$

$$= \underset{h\to 0}{\text{Lt}}\; h \frac{\sin (n-1)\frac{h}{2} \sin \frac{nh}{2}}{\sin \frac{h}{2}}$$

$$= \underset{h\to 0}{\text{Lt}}\; h \frac{\left\{\cos \frac{h}{2} - \cos (2n-1)\frac{h}{2}\right\}}{2 \sin \frac{h}{2}}$$

$$= \underset{h\to 0}{\text{Lt}} \left\{\cos \frac{h}{2} - \cos\left(\frac{\pi}{2} - \frac{h}{2}\right)\right\} \frac{\frac{h}{2}}{\sin \frac{h}{2}} = 1.$$

Ex. 3. Show that $\underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{r=n-1} \frac{n}{n^2+r^2}$ is $\frac{\pi}{4}$.

(B.Sc. 52 M and B.E. 47)

Series $\sum_{r=0}^{r=n-1} \frac{n}{n^2+r^2} = \sum_{r=0}^{r=n-1} \frac{1}{n} \frac{1}{1+r^2/n^2}$.

Consider $\int_0^1 \frac{dx}{1+x^2}$.

Putting $nh = 1$,

$$\int_0^1 \frac{dx}{1+x^2} = \underset{h\to 0}{\text{Lt}}\; h\left[\frac{1}{1} + \frac{1}{1+h^2} + \dots + \frac{1}{1+(n-1)^2h^2}\right]$$

where $nh = 1$

$$= \underset{n\to\infty}{\text{Lt}} \frac{1}{n}\left[\frac{1}{1} + \frac{1}{1+\frac{1}{n^2}} + \dots + \frac{1}{1+\left(\frac{n-1}{n}\right)^2}\right]$$

$$= \sum_{r=0}^{r=n-1} \frac{n}{n^2+r^2}.$$

But $\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \pi/4$. Hence the result.

Aliter.

$$\sum_{r=0}^{n=1} \frac{n}{n^2+r^2} = \frac{1}{n} \sum_{r=0}^{r=n-1} \frac{1}{1+r^2/n^2},$$

$$= \frac{1}{n} \sum_{r=0}^{r=n-1} f\left(\frac{r}{n}\right).$$

If $f\left(\frac{r}{n}\right)=\frac{1}{1+r^2/n^2}$,

$\therefore f(x)=\frac{1}{1+x^2}$.

By Cor. § 57·1,

$$\underset{n\to\infty}{\text{Lt}}\sum_{r=0}^{r=n-1}\frac{h}{n^2+r^2}=\int_0^1\frac{dx}{1+x^2}$$

$$=\left(\tan^{-1}x\right)_0^1=\frac{\pi}{4}.$$

Ex. 4. Find $\underset{n\to\infty}{\text{Lt}}\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right\}^{1/n}$.

Let the required limit be A.

$$\therefore \log A=\underset{n\to\infty}{\text{Lt}}\frac{1}{n}\left\{\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+\cdots+\log\left(1+\frac{n}{n}\right)\right\}$$

$$=\underset{n\to\infty}{\text{Lt}}\sum_1^n\log\left(1+\frac{r}{n}\right)$$

$$=\underset{n\to\infty}{\text{Lt}}\sum_1^n f\left(\frac{r}{n}\right)\text{ where } f\left(\frac{r}{n}\right)=\log\left(1+\frac{r}{n}\right)$$

$\therefore \log A=\int_0^1\log(1+x)\,dx$ by Cor. § 57·1

$$=\Big[x\log(1+x)\Big]_0^1-\int_0^1\frac{x}{1+x}\,dx$$

$$=\log 2-\int_0^1\left(1-\frac{1}{1+x}\right)dx$$

$$=\log 2-\Big[x-\log(1+x)\Big]_0^1=2\log 2-1$$

$$=\log\left(\frac{4}{e}\right)$$

$\therefore\quad A=4/e.$

§ 57·2. Geometrical Meaning of integration as summation.

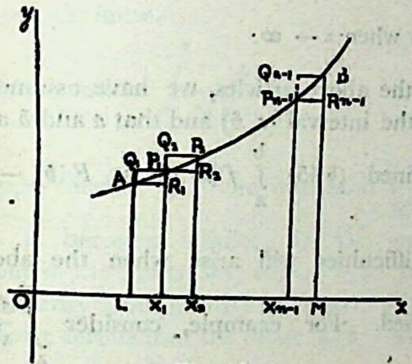

Fig. 54

Without loss of generality, we can take ξ_r of § 57·1 to coincide with x_{r-1} the lower end point of the corresponding sub-interval (x_{r-1}, x_r). These sub-intervals need not necessarily be equal.

Let $L, X_1, X_2 \ldots\ldots X_{n-1}, M$ be points on the x-axis with abscissae $a, x_1, x_2 \ldots\ldots x_{n-1}, b$ and let the corresponding ordinates of the graph $y = f(x)$ be LA, X_1P_1, X_2P_2 ...... $X_{n-1}P_{n-1}$, MB. Then their magnitudes are respectively $f(a), f(x_1), f(x_2) \ldots\ldots f(x_{n-1}), f(b)$.

The sum of the inner rectangles AR_1X_1L, $P_1R_2X_2X_1$ ...... $P_{n-1}R_{n-1}MX_{n-1} = f(a)(x_1 - a) + f(x_1)(x_2 - x_1) + \ldots\ldots + f(x_{n-1})(b - x_{n-1})$.

Let this sum be denoted by s. Let A be the area $ABML$ and S the sum of the outer rectangles $LQ_1P_1X_1$, $X_1Q_2P_2X_2$ ...... $X_{n-1}Q_{n-1}BM$. It is evident that $S > A > s$.

Besides, $A - s =$ sum of the areas AR_1P_1, $P_1R_2P_2$ ...... $P_{n-1}R_{n-1}B$. If η be the length of the greatest sub-interval, then $A - s < \eta\,[R_1P_1 + R_2P_2 + \ldots\ldots + R_{n-1}B] < \eta\,[f(b) - f(a)]$.

When the number of sub-intervals increases indefinitely, $\eta \to 0$ and $A - s \to 0$ as $f(b) - f(a)$ is finite.

Thus the area $ALMB$ is the limiting value of the sum of the inner rectangles. By a similar argument, the sum of the outer rectangles also tends to the area $ALMP$. It will be shown (*vide* § 58·1) that area $ALMB = \int_a^b f(x)\,dx$. Thus

$s = \sum_{r=1}^{r=n-1} f(x_{r-1})(x_r - x_{r-1})$ and $S = \sum_{r=1}^{r=n} f(x_r)(x_r - x_{r-1})$, both tend to $\int_a^b (x)\, dx$ when $x \to \infty$.

Note.—In the above articles, we have assumed that $f(x)$ is continuous in the interval (a, b) and that a and b are both finite. We have defined (§ 45) $\int_a^b f(x)\, dx$ as $F(b) - F(a)$, where $F'(x) = f(x)$.

Certain difficulties will arise when the above conditions are not satisfied. For example, consider $\int_a^\infty \frac{dx}{x^2}$ $(a > 0)$ and $\int_0^\infty \frac{dx}{a^2 + x^2}$. In these integrals, the upper limit of integration is not finite. In the general case,

$$\int_a^b f(x)\, dx = F(b) - F(a)\;;$$

if, as $b \to \infty$, $F(b) \to$ a finite limit l, then

$$l - F(a) \text{ is said to be the value of } \int_a^\infty f(x)\, dx.$$

(Here $\int_a^x f(x)\, dx$ is said to converge as $x \to \infty$.)

Taking $\int_a^b \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_a^b = \frac{1}{a} - \frac{1}{b}$.

As $b \to \infty$, $\frac{1}{b} \to 0$. Hence $\int_a^\infty \frac{dx}{x^2} = \frac{1}{a}$.

Similarly, $\int_0^b \frac{dx}{a^2 + x^2} = \frac{1}{a}\left[\tan^{-1}\frac{x}{a}\right]_0^b = \frac{1}{a}\tan^{-1}\frac{b}{a}$.

As $b \to \infty$, $\tan^{-1}\frac{b}{a} \to \frac{\pi}{2}$. Hence $\int_0^\infty \frac{dx}{a^2 + x^2} = \frac{\pi}{2a}$.

There arise, in certain examples, discontinuities for the integrand in the range of integration considered while the limits of integration remain finite.

For example, consider $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$.

At the upper limit $x=1$, the integrand $\frac{1}{\sqrt{1-x^2}}$ is discontinuous as it becomes infinite. In the general case, let $f(x) \to \infty$ at one of the limits, say, at $x=b$, so that $f(b) \to \infty$. Let ϵ be an arbitrarily small positive quantity. $f(x)$ is continuous and therefore finite throughout the range $(a, b-\epsilon)$. The integral $\int_a^b f(x)\,dx$ is defined as $\underset{\epsilon \to 0}{\text{Lt}} \int_a^{b-\epsilon} f(x)\,dx$, provided this latter limit exists. Similarly we can deal with an infinity for the integrand at the lower limit. If $f(a) \to \infty$, $\int_a^b f(x)\,dx = \underset{\epsilon \to 0}{\text{Lt}} \int_{a+\epsilon}^b f(x)\,dx$, provided this limit exists.

Now, in the example considered, $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$, we have to find

$$\underset{\epsilon \to 0}{\text{Lt}} \int_0^{1-\epsilon} \frac{dx}{\sqrt{1-x^2}} = \underset{\epsilon \to 0}{\text{Lt}} \Big[\sin^{-1} x\Big]_0^{1-\epsilon}$$

$$= \underset{\epsilon \to 0}{\text{Lt}} \sin^{-1}(1-\epsilon) = \sin^{-1} 1 = \frac{\pi}{2}.$$

Hence $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$.

$\int_0^1 \frac{dx}{\sqrt{x}}$ is an example where $\frac{1}{\sqrt{x}}$ becomes infinite at $x=0$, the lower limit. Hence $\int_0^1 \frac{dx}{\sqrt{x}}\,dx = \underset{\epsilon \to 0}{\text{Lt}} \int_\epsilon^1 \frac{dx}{\sqrt{x}}$, if this limit exists.

$$\underset{\epsilon \to 0}{\text{Lt}} \int_{\epsilon}^{1} \frac{dx}{\sqrt{x}} = \underset{\epsilon \to 0}{\text{Lt}} \Big[2\sqrt{x} \Big]_{\epsilon}^{1} = 2 \underset{\epsilon \to 0}{\text{Lt}} (1 - \sqrt{\epsilon}) = 2.$$

Hence $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$ exists and is 2.

Exercises LXVIII.

Obtain from the definition of a definite integral as the limit of a sum :

1. $\displaystyle\int_1^2 x^2\, dx.$ (B.A. Sub. 42). 2. $\displaystyle\int_1^2 x^3\, dx.$

3. $\displaystyle\int_a^b e^x\, dx.$ 4. $\displaystyle\int_a^b \cos x\, dx.$

5. Using the definition of integration as a process of summation, show that

(*i*) $\displaystyle\underset{n\to\infty}{\text{Lt}} \left[\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{3n}\right] = \log 3.$ (B.Sc. 53 M)

(*ii*) $\displaystyle\underset{n\to\infty}{\text{Lt}} \left[\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}\right] = \log 2.$ (B.A. 47 M ; B.Sc. 59)

(*iii*) $\displaystyle\underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{r=n} \frac{1}{\sqrt{4n^2 - r^2}} = \frac{\pi}{6}.$ (B.A. 54 M)

(*iv*) $\displaystyle\underset{n\to\infty}{\text{Lt}} \frac{1}{n^3}\left[1 + 4 + 9 + \dots + n^2\right] = \frac{1}{3}.$

(*v*) $\displaystyle\underset{n\to\infty}{\text{Lt}} \left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \dots + \frac{1}{8n}\right] = \frac{3}{8}.$

(*vi*) $\displaystyle\underset{n\to\infty}{\text{Lt}} \frac{1}{n}\left[\sin^{2k}\frac{\pi}{2n} + \sin^{2k}\frac{2\pi}{2n} + \dots + \sin^{2k}\frac{\pi}{2}\right] = \frac{2k!}{2^k (k!)^2}$

(*vii*) $\displaystyle\underset{n\to\infty}{\text{Lt}} \left[\frac{1}{1+n^3} + \frac{4}{8+n^3} + \dots + \frac{1}{2n}\right] = \frac{\log 2}{3}.$

(*viii*) $\displaystyle\underset{n\to\infty}{\text{Lt}} \left[\left(1 + \frac{1}{n^2}\right)\left(1 + \frac{2^2}{n^2}\right)\dots\left(1 + \frac{n^2}{n^2}\right)\right]^{1/n} = 2e^{\frac{\pi-4}{2}}.$

(ix) $\underset{n\to\infty}{\text{Lt}} \dfrac{1}{3n+1} + \dfrac{1}{3n+2} + \dots + \dfrac{1}{3n+n} = \log\frac{4}{3}.$

(x) $\underset{n\to\infty}{\text{Lt}} \dfrac{[\sqrt{n+1} + \sqrt{n+2} + \dots + \sqrt{n+n}]}{n\sqrt{n}}$

$$= \tfrac{2}{3}(2\sqrt{3} - 1).$$

(xi) $\underset{n\to\infty}{\text{Lt}} \left\{\dfrac{n^{1/2}}{n^{3/2}} + \dfrac{n^{1/2}}{(n+3)^{3/2}} + \dfrac{n^{1/2}}{(n+6)^{3/2}} + \dots\right.$

$$\left. + \frac{n^{1/2}}{\{n+3(n-1)\}^{3/2}}\right\} = \frac{1}{3}.$$

(xii) $\underset{n\to\infty}{\text{Lt}} \left[\dfrac{1}{n^2}\sqrt{n^2-1^2} + \dfrac{1}{n^2}\sqrt{n^2-2^2} + \dots \text{ to } n \text{ terms}\right]$

$$= \frac{\pi}{4}. \qquad \text{(B.E. 55)}$$

(xiii) Find $\underset{n\to\infty}{\text{Lt}} \displaystyle\sum_{r=0}^{r=n-1} \frac{1}{\sqrt{n^2-r^2}}.$ (B.Sc. 56 M)

(xiv) $\underset{n\to\infty}{\text{Lt}} \left[\dfrac{1}{n^2+1^2} + \dfrac{2}{n^2+2^2} + \dfrac{3}{n^2+3^2} + \dots\right.$

$$\left. + \frac{n}{n^2+n^2}\right] = \frac{1}{2}\log 2.$$

6. Show that $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x}\,dx > \int_{\pi/2}^{\pi} \frac{\sin x}{x}\,dx.$ (B.Sc. 51 M)

7. Defining $\log x = \displaystyle\int_1^x \frac{dz}{z}$,

prove that $\dfrac{x}{1+x} < \log(1+x) < x$ where $x > 0$.

8. Evaluate :

(a) $\displaystyle\int_1^{\infty} \frac{dx}{x^3}.$ (b) $\displaystyle\int_0^1 \frac{dx}{x^{1/3}}.$ (c) $\displaystyle\int_{-1}^{+1} \frac{dx}{x^{1/3}}.$

9. Does $\displaystyle\int_a^b \frac{dx}{(a-x)^2}$ exist ?

CHAPTER XIII

GEOMETRICAL APPLICATIONS OF INTEGRATION

§ 58·1. Areas under plane curves : cartesian coordintes.

We shall find a formula for the area bounded by the arc of the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the portion of the x-axis between them.

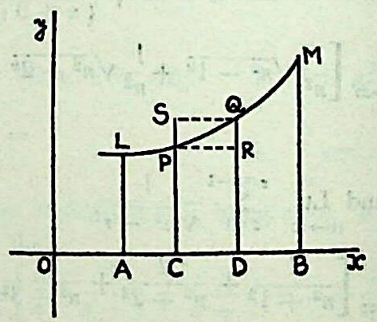

Fig. 55

For definiteness, let us suppose that $a < b$. In the figure, the graph of $y = f(x)$ is $LPQM$. Let $OA = a$ and $OB = b$. Then the ordinates AL and BM are $y = f(a)$ and $y = f(b)$ respectively.

We shall prove that the area bounded by the arc LM, the ordinates AL and BM and the portion AB of the x-axis is $\int_a^b f(x)\, dx$.

Let P be any point (x, y) on the curve and Q a neighbouring point $(x + \Delta x, y + \Delta y)$ on it. Draw the ordinates PC and QD and draw PR and QS perpendicular to QD and CP respectively.

Let A represent the area bounded by the arc LP, the ordinates AL, CP and the portion AC of the x-axis. Then the area $ALQD$ can be represented by $A + \Delta A$ so that the area $CPQD$ is ΔA. From the figure, it can be seen that the area $CPQD$ is greater than the inner rectangle $CPRD$ and is less than the outer rectangle $CSQD$.

Rectangle $CPRD = CP \cdot CD = y\ \Delta x$ and rectangle $CSQD = DQ \cdot CD = (y + \Delta y)\ \Delta x$.

$\therefore\ \Delta A > y\ \Delta x$ but $< (y + \Delta y)\ \Delta x$.

$$\therefore\ y < \frac{\Delta A}{\Delta x} < (y + \Delta y).$$

Proceeding to the limit when $\Delta x \to 0$,

$$\frac{\Delta A}{\Delta x} \to \frac{dA}{dx} \text{ and } y + \Delta y \to y.$$

$\therefore\ \dfrac{dA}{dx}$ lies between y and a quantity which tends to y in the limit. Hence $\dfrac{dA}{dx} = y$.

$\therefore\ A = \int y dx + C$, where C is the constant of integration, *i.e.*, $A = \int f(x)\ dx + C$.

Let us denote $\int f(x)\ dx$ by $F(x)$. Then

$$A = F(x) + C.$$

When $x = a$, $A = 0$ as A is, by definition, area $ALPC$.

$$\therefore\quad 0 = F(a) + C. \qquad (1)$$

When $x = b$, $A =$ area $ALMB$ which is sought.

$\therefore$ **The required area ALMB** $= F(b) + C$

$= F(b) - F(a)$ on substituting for C from (1)

$$= \Big[F(x)\Big]_a^b = \int_a^b \mathbf{f(x)\,dx}.$$

Note.—(*i*) There is a point in the above proof which deserves our notice. In the curve we have drawn, y increases with x. If y decreases as x increases, as in the figure below, the same formula for the area holds good.

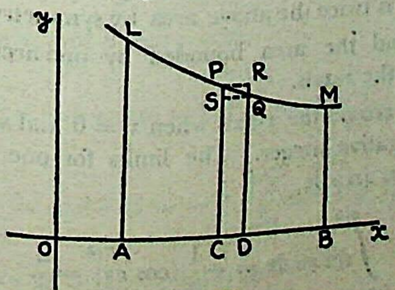

Fig. 56

With the same notation, we find here that

$$\Delta A < y\,\Delta x \text{ and } > (y + \Delta y)\,\Delta x.$$

$$\therefore\ y\,\Delta x > \Delta A > (y + \Delta y)\,\Delta x.$$

$$\therefore\ y > \frac{\Delta A}{\Delta x} > (y + \Delta y).$$

These inequalities are reversed in the case of an increasing function. But, in the limit, when $\Delta x \to 0, \dfrac{dA}{dx} = y$ and the rest of the proof is the same as before. Whether y increases or decreases with x, the above formula holds good.

(*ii*) One other point also deserves special mention. When part of the area is below the x-axis, the corresponding ordinates are negative and Δx being taken to be positive, the area will be negative.

(*iii*) The formula for the area under a curve, the y-axis and the lines (abscissae) $y = c$, $y = d$ is $\int_c^d x\,dy$ by a similar argument.

Examples.

Ex. 1. Find the area bounded by the curve $y^2 = 4ax$, the x-axis and the ordinate $x = h$.

The curve is the parabola which, we know, passes through the origin. The limits for the area in question are 0 and h. Hence the required area is

$$\int_0^h \sqrt{4ax}\,dx = \frac{4\sqrt{a}}{3}\,\left(x^{3/2}\right)_0^h = \frac{4h\sqrt{ah}}{3}.$$

(The area bounded by the above parabola and the double ordinate $x = h$ is twice the above area by symmetry.)

Ex. 2. Find the area bounded by one arch of the curve $y = \sin ax$ and the x-axis.

The curve crosses the x-axis when $x = 0$ and $n\pi/a$, where n is a positive or negative integer. The limits for one arch are 0 and π/a. Hence the area is

$$\int_0^{\pi/a} \sin ax\,dx = -\frac{1}{a}\,\left(\cos ax\right)_0^{\pi/a} = \frac{2}{a}.$$

Ex. 3. Find the area bounded by one arch of the cycloid $x = a(\theta - \sin\theta)$; $y = a(1 - \cos\theta)$ and its base.

(For the tracing of the curve, *vide* page 230.)

As the point P describes one arch, the parameter θ varies from 0 to 2π.

$$\therefore \text{ Area required} = \int_0^{2\pi} y \frac{dx}{d\theta} d\theta$$

$$= \int_0^{2\pi} a(1 - \cos\theta) \times a(1 - \cos\theta)\, d\theta$$

$$= a^2 \int_0^{2\pi} (1 - \cos\theta)^2 d\theta = a^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= a^2 \int_0^{2\pi} \left(1 + \frac{1 + \cos 2\theta}{2}\right) d\theta \text{ as } \int_0^{2\pi} \cos\theta\, d\theta = [\sin\theta]_0^{2\pi} = 0;$$

$$= 3\pi a^2 \text{ as } \int_0^{2\pi} \cos 2\theta\, d\theta = \left[\frac{\sin 2\theta}{2}\right]_0^{2\pi} = 0.$$

Ex. 4. Find the area of a loop of the curve

$$y^2 = x^2 \frac{a + x}{a - x}.$$

(For the tracing of the curve, *vide* similar Ex. 4 on page 221.)

The limits for the loop are $-a$ and 0.

As the curve is symmetrical about the x-axis, the area of the loop = twice the area of the loop above the x-axis

$$= 2\int_0^{-a} y\, dx = 2\int_0^{-a} x \sqrt{\frac{a + x}{a - x}}\, dx.$$

To integrate, put $x = a\cos 2\theta$.

The integral reduces to $-2\displaystyle\int_{\pi/4}^{\pi/2} a^2 \cos 2\theta \frac{\cos\theta}{\sin\theta} \sin 2\theta\, d\theta$

$$= -4a^2 \int_{\pi/4}^{\pi/2} \cos 2\theta \cos^2\theta\, d\theta = -2a^2 \int_{\pi/4}^{\pi/2} \cos 2\theta\,(1 + \cos 2\theta)\, d\theta$$

$$= -a^2 \int_{\pi/4}^{\pi/2} (2\cos 2\theta + 1 + \cos 4\theta)\, d\theta$$

$$= -a^2\left[\sin 2\theta + \theta + \frac{\sin 4\theta}{4}\right]_{\pi/4}^{\pi/2} = -a^2\left[\frac{\pi}{2} - 1 - \frac{\pi}{4}\right]$$

$$= a^2\left(1 - \frac{\pi}{4}\right).$$

Hence the area required is $2a^2\left(1 - \frac{\pi}{4}\right)$.

Exercises LXIX.

1. Find the area bounded by the curve $x^2 = 4y$, the x-axis and $x = 2$.

2. Find the area of an elliptic quadrant.

3. Find the area bounded by an arch of the curve $y = \cos 2x$ and the x-axis.

4. Find the areas of the portions into which the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is divided by the line $x = c$.
(B.A. 49 ; B.Sc. 52 T.U.)

5. Find the area included between the curve $ay^2 = x^3$, the x-axis and the ordinate $x = a$.

6. Find the area bounded by the x- and y-axes and the curve (i) $x^{2/3} + y^{2/3} = a^{2/3}$, (ii) $\sqrt{x} + \sqrt{y} = 1$.

7. Find the area bounded by $y = 5x - x^2 - 4$ and the x-axis.

8. Find the area of the loop of the curve

(i) $y^2 = \frac{x^2(1+x)}{1-x}$. (B.A. 46 ; B.Sc. 51 T.U.)

(ii) $y^2 = x^4(x+2)$. (B.A. 37)

(iii) $2y^2 = (1-x)(1+2x)^2$. (B.A. 38)

(iv) $4y^2 = (x-5)^2(x-1)$. (B.Sc. 51 T.U.)

(v) $ay^2 = x^3 + ax^2$. (B.Sc. 53 T.U.)

(vi) $y^2 = (x+1)(x-2)^2$. (B.A. 53 M)

(vii) $ay^2 = (x-a)(x-5a)^2$.

9. Show that there is a loop of the curve $y^2 = x^2(4-x^2)$ between $x = 0$ and $x = 2$ and find the area of the loop.
(B.A. Sub. 49)

10. Trace the curve $ay^2 = x^2(a-x)$ and show that the area of its loop is $8a^2/15$.

11. Find the area bounded by $y = a \sin x$ and $y = a \cos x$ between the two successive intersections. (B.Sc. Anc. 59)

12. If A is a vertex, O the centre and P any point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, prove that $x = a \cosh\left(\dfrac{2S}{ab}\right)$, $y = b \sinh\left(\dfrac{2S}{ab}\right)$ where S is the sectorial area AOP. (B.Sc. 53)

§ 58·2. Area of a closed curve.

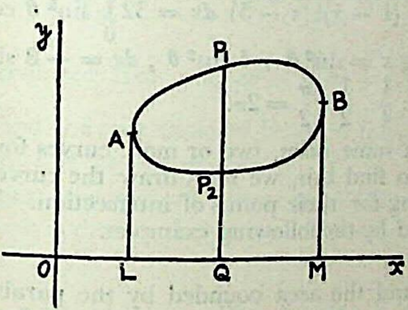

Fig. 57

Let AL and BM be the tangents to the closed curve parallel to the y-axis. Let an intermediate ordinate meet the curve in two points P_1 and P_2 where P_1 is (x, y_1) and P_2 is (x, y_2). Let $y_1 > y_2$. Denoting OL and OM by a and b respectively,

area $LAP_1BM = \int_a^b y_1 \, dx$ and area $LAP_2BM = \int_a^b y_2 \, dx$.

By subtraction, we get the area of the closed curve to be $\int_a^b (y_1 - y_2) \, dx$.

This integral gives the area whether the x-axis cuts the curve or not. The values y_1 and y_2 corresponding to any value of x are found by solving the equation of the curve as a quadratic in y.

Example. Find the area of the ellipse

$$x^2 + 4y^2 - 6x + 8y + 9 = 0.$$

Writing this as a quadratic in y,

$$4y^2 + 8y + x^2 - 6x + 9 = 0.$$

If y_1 and y_2 be the roots,

$$y_1 + y_2 = -2 \text{ and } y_1 y_2 = \frac{x^2 - 6x + 9}{4}.$$

c—21

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Hence $y_1 - y_2 = \sqrt{(y_1+y_2)^2 - 4y_1y_2}$
$= \sqrt{4 - (x^2 - 6x + 9)} = \sqrt{6x - x^2 - 5}$
$= \sqrt{(1-x)(x-5)}$.

The two values y_1 and y_2 are equal when $x = 1$ and $x = 5$. These are the abscissae of the points at which the tangents are parallel to the y-axis.

Hence the area of the ellipse $= \int_1^5 (y_1 - y_2)\, dx$

$$= \int_1^5 \sqrt{(1-x)(x-5)}\, dx = 32 \int_0^{5/2} \sin^2\theta \cos^2\theta\, d\theta$$

(on putting $x = \sin^2\theta + 5\cos^2\theta$; $dx = -8 \sin\theta\cos\theta\, d\theta$) ;

$$= 32 \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 2\pi.$$

§ **58·3.** In some cases, two or more curves form the contour of an area. To find this, we must draw the curves and find the limits by solving for their points of intersection. The method is best exemplified by the following examples.

Examples.

Ex. 1. Find the area bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4by$. $y^2 = 4ax$ has Ox as its axis and $x^2 = 4by$, Oy as its axis. To find the abscissae of their points of intersection, eliminate y between the equations. We get $x^4 = 16b^2 \cdot 4ax$. Hence $x = 0$ or $4a^{1/3}b^{2/3}$; $x = 0$ corresponds to O the origin. $x = 4a^{1/3}b^{2/3}$ corresponds to A the other point of intersection.

Area required $=$ Area $OBAL -$ Area $OCAL$.

Area $OBAL = \int_0^{4a^{1/3}b^{2/3}} y\, dx$ where $y^2 = 4ax$;

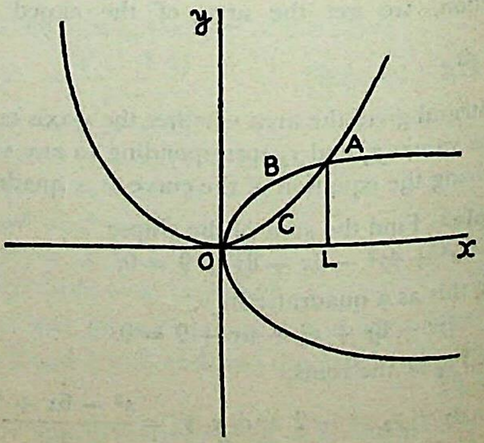

Fig. 58

$$= 2\sqrt{a}\int_0^{4a^{1/3}b^{2/3}} \sqrt{x}\, dx = \frac{4\sqrt{a}}{3}\Big[x^{3/2}\Big]_0^{4a^{1/3}b^{2/3}} = \frac{32}{3}ab.$$

Area $OCAL = \int_0^{4a^{1/3}b^{2/3}} y\,dx$ where $x^2 = 4by$;

$$= \frac{1}{4b}\int_0^{4a^{1/3}b^{2/3}} x^2\,dx = \frac{1}{12b}\Big[x^3\Big]_0^{4a^{1/3}b^{2/3}} = \frac{16}{3}ab.$$

$\therefore$ The required area $= \frac{32}{3}ab - \frac{16}{3}ab = \frac{16ab}{3}.$

Ex. 2. Find the area enclosed between the parabola $y = x^2$ and the straight line $2x - y + 3 = 0$.

The straight line cuts off intercepts $-3/2$ and 3 from the axes and its graph is drawn. Let it cut the parabola at two points.

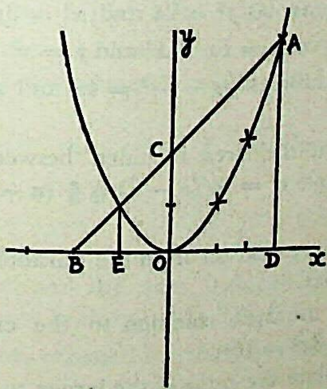

Fig. 59

The abcissae of these points, *viz.*, OD and OE are got by eliminating y between the two equations and solving the resulting quadratic. From the straight line equation, $y = 2x + 3$. Putting this in $y = x^2$, we have $x^2 = 2x + 3$, *i.e.*, $x^2 - 2x - 3 = 0$.

$\therefore$ $x = 3$ or -1. Hence $OD = 3$ and $OE = -1$.

Area required $= \int_{-1}^{3} (y_1 - y_2)\,dx$ where y_1 is the ordinate of the straight line corresponding to x, i.e., $y_1 = 2x + 3$ and y_2 the ordinate of $y = x^2$, i.e., $y_2 = x^2$ by § 58·2.

$$\text{Hence the area} = \int_{-1}^{3} (2x + 3 - x^2)\, dx = \left(x^2 + 3x - \frac{x^3}{3}\right)_{-1}^{3}$$

$$= 9 + 9 - 9 - (1 - 3 + \tfrac{1}{3}) = 10\tfrac{2}{3}.$$

Exercises LXX.

1. Find the area bounded by

(i) the semi-cubical parabola $y^2 = x^3$, the y-axis, the lines $y = 1$ and $y = 2$.

(ii) the curves $y = \sin x, y = \cos x$ and the y-axis.

(iii) the curves $y = x^3$ and $y^2 = 9x$.

(iv) the parabola $y^2 = 4x$ and the straight line

$$2x - 3y + 4 = 0. \qquad \text{(B.A. 31)}$$

(v) the parabola $x^2 = 4ay$ and the curve $y = \dfrac{8a^3}{x^2 + 4a^2}$.

(B.A. 52 S)

(vi) the parabolas $y^2 = 2x$ and $x^2 = 3y$. (B.A. 40 S)

(vii) the curves $y = 3x - x^2$ and $y = x^2 - x$. (B.A. 46 S)

(viii) the parabolas $(y - 1)^2 = 4x$ and $x^2 = 4(y - 1)$.

(B.Sc. Sub. 55)

2. Show that the area included between the parabolas $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$ is $\frac{8}{3}(a + b)\sqrt{ab}$.

(B.Sc. 52 T.U.)

3. Find the area cut off from the parabola $y^2 = 4ax$ by the straight line $y = mx$.

4. (i) Find the area common to the circle $x^2 + y^2 = 25$ and the parabola $3x^2 = 16y$. (B.E. 51)

(ii) Calculate the ratio of the larger to the smaller of the two finite areas bounded by the curves

$$x^2 + y^2 = 25 \text{ and } x^2 = 4y - 7. \qquad \text{(B.Sc. 32 M)}$$

(iii) Find the area common to the two curves $y^2 = 4ax$ and $x^2 + y^2 = 8ax$. (B.Sc. 56 M)

5. Find the area common to the two ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

6. Find the area between the curve $y = \dfrac{x}{1 + x^2}$ and the line $4y = x$.

7. In the curve $y^2(a-x)=x^2(a+x)$, show that the area between the curve and the asymptote exceeds the area of the loop of the curve by πa^2. (B.Sc. 52 T.U.)

(*Vide* worked out Ex. 4, Page 221)

8. Show that the larger of the two areas into which the circle $x^2+y^2=64a^2$ is divided by the parabola $y^2=12ax$ is

$$\frac{16a^2}{3}(8\pi-\sqrt{3}).$$ (B.Sc. 53 M)

9. Give a rough sketch of the curves $ay^2=x^3$ and $y^2=a(2a-x)$ and show that the area enclosed by them can be expressed in the form

$$2\int_0^a\left(\frac{2a^2-y^2}{a}-a^{1/3}y^{2/3}\right)dy \text{ and is equal to } \frac{32}{15}a^2.$$

(Pilani 51 Eng.)

10. Find the area of the ellipse

$$3x^2-10xy+10y^2+8x-20y+10=0.$$

11. Find the area enclosed between the parabola $x^2=2y$, the tangent at the point whose abscissa is 3 and the x-axis. (B.A. 55 M)

12. Prove that the area of the loop of the curve $y^2(a+x)=x^2(a-x)$ is $a^2\left(2-\frac{\pi}{2}\right)$.

13. Trace the curve $y^2(a+x)=x^2(3a-x)$ and prove that the area of the loop and the area included between the curve and the asymptote are both equal to $3a^2\sqrt{3}$.

14. Find the area enclosed between the curve $y(x^2+4a^2)=8a^3$ and its asymptote. (B.E. 55 M).

§ 58·4. Areas in polar coordinates.

We propose to find a formula for the area bounded by the curve whose polar equation is $r=f(\theta)$ and two radii vectors OA and OB.

Let $x\hat{O}A$ and $x\hat{O}B$ be respectively α and β.

Let P be a point (r, θ) on the curve and P' a neighbouring point $(r+\Delta r, \theta+\Delta\theta)$ on it. If we denote the area AOP by A, then the area denoted by AOP' is $A+\Delta A$ so that area POP' is ΔA. Let the circle centre O and radius OP cut OP' at M and the circle centre O and radius OP' cut OP produced at N.

Area POP' lies between the areas of the circular sectors OPM and $OP'N$, *i.e.*, $\frac{1}{2} r^2 \Delta\theta$ and $\frac{1}{2}(r+\Delta r)^2 \Delta\theta$.

$$\therefore \tfrac{1}{2}(r+\Delta r)^2 \Delta\theta > \Delta A > \tfrac{1}{2} r^2 \Delta\theta.$$

Dividing by $\Delta\theta$ and proceeding to the limit as $\Delta\theta \to 0$, we have $\dfrac{dA}{d\theta} = \dfrac{1}{2} r^2$.

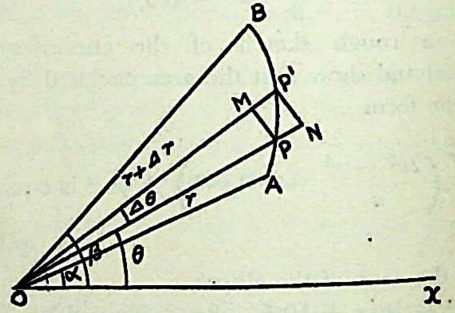

Fig. 60

$$\therefore A = \tfrac{1}{2}\int r^2\, d\theta + C = F(\theta) + C$$

where $F(\theta) = \frac{1}{2}\int r^2\, d\theta$.

Putting $\theta = \alpha$, $A = 0$. $\therefore 0 = F(\alpha) + C$. (i)

Putting $\theta = \beta$, $A =$ area $OAB = F(\beta) + C$. (ii)

$\therefore$ By subtraction, area $OAB = F(\beta) - F(\alpha)$

$$= \Big[F(\theta)\Big]_{\alpha}^{\beta} = \tfrac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta.$$

Examples.

Ex. 1. Find the area of the cardioid $r = a(1+\cos\theta)$ (*vide* Ex. 8, Page 228, for the tracing of the curve). Since the curve is symmetrical about the initial line, the area

$$= 2\int_0^{\pi} \tfrac{1}{2} r^2\, d\theta = a^2 \int_0^{\pi} (1+\cos\theta)^2\, d\theta = 4a^2 \int_0^{\pi} \cos^4 \theta/2\, d\theta$$

$$= 8a^2 \int_0^{\pi/2} \cos^4 \phi\, d\phi \text{ (on putting } \theta/2 = \phi)$$

$$= 8a^2 \cdot \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi a^2}{2}.$$

Ex. 2. Find the entire area of the lemniscate of Bernouilli $r^2 = a^2 \cos 2\theta$.

The area consists of two loops and each loop is symmetrical about the initial line. (*Vide* Ex. 9, Page 229 for the tracing of the curve.)

The area required $= 4 \times$ the area of one half loop of the curve above the initial line

$$= 4 \int_0^{\pi/4} \tfrac{1}{2} r^2 \, d\theta = 2a^2 \int_0^{\pi/4} \cos 2\theta \, d\theta = a^2 \Big[\sin 2\theta\Big]_0^{\pi/4} = a^2.$$

Exercises LXXI.

1. Find the area between the equiangular spiral $r = ae^{\theta \cot \alpha}$ and two radii vectors of lengths r_1 and r_2.

2. Find the area of the circle $r = 2a \cos \theta$ by integration.

3. Find the area of one loop of the curve $r = a \cos 3\theta$.

4. Find the area enclosed within the curve
$r = 4(1 + \cos \theta)$. (B.Sc. 53 T.U.)

5. Find the area of the parts into which the curve $r = a(1 + \cos \theta)$ is divided by a line through the pole perpendicular to the initial line.

6. Find the area of the curve $r = 3 + 2 \cos \theta$.
(B.Sc. 53 Os.U.) (*Vide* Ex. 8, Page 227)

7. Draw the curve $r = 2 + 3 \cos \theta$ and find the area enclosed between the two loops. (B.Sc. 51 T.U.)

8. Show that the area of a loop of the curve $r = a \sin 3\theta$ is $\pi a^2/12$. (B.Sc. 51 M)

(*Vide* Ex. 6, Page 225, for the tracing of the curve.)

9. Find the area common to the circles $r = a\sqrt{2}$ and $r = 2a \cos \theta$.

10. Find the area situated outside the circle $r = 2a \cos \theta$ and inside the cardioid $r = a[1 + \cos \theta]$. (B.Sc. 52 M ; B.Sc. Sub. 53)

11. Prove that the sum of the areas of the two loops of the limacon $r = a + b \cos \theta \; (b > a)$ is $\dfrac{\pi}{2}(2a^2 + b^2)$.

12. Show that the area included by the parabola $r \cos^2 \dfrac{\theta}{2} = a$ and the focal vectors of lengths a and r is $\frac{1}{3}\sqrt{a(r-a)}\,.\,(2a+r)$. (B.Sc. 39)

§ 59. Approximate integration.

We now give two rules for evaluating $\int_a^b f(x)\,dx$ approximately. These are useful when integration is impossible in terms of known functions.

§ 59·1. Trapezoidal Rule.

The exact value of $\int_a^b f(x)\,dx$ gives the measure of the area bounded by the curve $y = f(x)$, $x = a$, $x = b$ and portion of the x-axis. We shall dissect this area into trapezoids and by adding their areas, evaluate the total area approximately.

Divide the segment $b - a$ on Ox into n equal parts, each of length Δx. Let us denote the abscissae of the successive points of division by

$$x_0\ (= a),\ x_1,\ x_2,\ \ldots\ldots,\ x_n\ (= b).$$

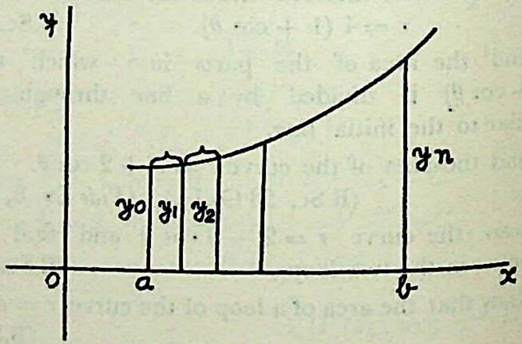

Fig. 61

Let the corresponding ordinates be

$$y_0 = f(x_0),\quad y_1 = f(x_1),\quad \ldots\ldots\quad y_n = f(x_n).$$

By joining the extremities of consecutive ordinates, we form trapezoids. As the area of a trapezium is one half the product of the sum of the parallel sides and the altitude, we get

area of the first trapezium $= \frac{1}{2}(y_0 + y_1)\,\Delta x$,

area of the second trapezium $= \frac{1}{2}(y_1 + y_2)\,\Delta x$,

....

area of the last (nth) trapezium $= \frac{1}{2}(y_{n-1} + y_n)\,\Delta x$.

Adding, the sum of the trapezoids

$$= \Delta x\,\{\tfrac{1}{2}y_0 + y_1 + y_2 + \ldots\ldots + y_{n-1} + \tfrac{1}{2}y_n\}.$$

This expression is an approximation to the required area.

Example. Calculate $\int_1^6 x^2\,dx$ by the trapezoidal rule.

We shall, for illustration, divide the interval (1, 6) into five equal parts each of length 1. $\therefore\ \Delta x = \dfrac{6-1}{5} = 1.$

When $x = 1, 2, 3, 4, 5, 6$, the corresponding ordinates y are 1, 4, 9, 16, 25, 36 as $y = x^2$.

By the trapezoidal rule, the area

$$= (\tfrac{1}{2}.1 + 4 + 9 + 16 + 25 + \tfrac{1}{2}.36) \times 1 = 72{\cdot}5.$$

The exact area $= \displaystyle\int_1^6 x^2\,dx = \left[\frac{x^3}{3}\right]_1^6 = 71\tfrac{2}{3}.$

Hence the error in the approximation is roughly 1 per cent.

§ 59·2. **Simpson's Rule.** A closer approximation to the area than the trapezoidal rule is furnished by what is known as Simpson's rule. Here we join the extremities of successive ordinates by arcs of parabolas and sum up the areas under these arcs.

Let the three points A, B, C on the given curve have ordinates y_1, y_2, y_3 whose abscissae are $-h$, 0, h respectively. Let us take h to be small. If we pass a parabola through these points with its axis

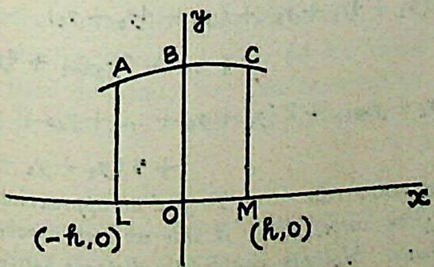

Fig. 62

parallel to the y-axis, its equation is of the form $y = a + bx + cx^2$. The values of a, b, c are determined by expressing that $A\ (-h, y_1)$, $B\ (0, y_2)$ and $C\ (h, y_3)$ lie on the curve.

$$\therefore\ y_1 = a - bh + ch^2\ ;\quad y_2 = a\ ;\quad y_3 = a + bh + ch^2. \qquad (1)$$

The area bounded by this parabola, the portion of the x-axis and the ordinates $x = \pm h$ is

$$\int_{-h}^{+h} (a + bx + cx^2)\, dx = \left(ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right)_{-h}^{h}$$

$$= 2\left(ah + \frac{ch^3}{3}\right) = \frac{2h}{3}(3a + ch^2).$$

From (1), $y_1 + y_3 = 2(a + ch^2)$.

$\therefore \; y_1 + 4y_2 + y_3 = 6a + 2ch^2 = 2(3a + ch^2)$.

Hence, the area under the parabolic arc

$$= \frac{h}{3}(y_1 + 4y_2 + y_3). \qquad (2)$$

This area is a close approximation to the area $ALMC$.

Let us take an odd number, *i.e.*, $2n + 1$ equidistant ordinates of the given curve and let the successive ordinates be denoted by $y_1, y_2 \ldots y_{2n+1}$.

The area under a parabolic arc passing through the extremities of the ordinates y_1, y_2 and $y_3 = \frac{h}{3}(y_1 + 4y_2 + y_3)$ by (2).

Similarly the area under a parabolic arc passing through the extremities of the ordinates $y_3, y_4, y_5 = \frac{h}{3}(y_3 + 4y_4 + y_5)$ and so on.

The total area under several parabolic arcs so drawn

$$= \frac{h}{3}\{(y_1 + 4y_2 + y_3) + (y_3 + 4y_4 + y_5) + \ldots + (y_{2n-1} + 4y_{2n} + y_{2n+1})\}$$

$$= \frac{h}{3}\{y_1 + y_{2n+1} + 2(y_3 + y_5 + \ldots + y_{2n-1}) + 4(y_2 + y_4 + \ldots + y_{2n})\}.$$

Thus Simpson's rule is : *To find an approximate value of the area under a given curve, divide it into an even number of strips by equidistant ordinates. Multiply one-third the distance between two consecutive ordinates by the sum of the first, the last, twice the sum of the other odd ordinates and four times the sum of all the even ordinates.*

If the curve crosses the x-axis at one or both ends of the required area, one or both of the extreme ordinates must be taken as zero.

Example. Compute $\int_0^8 \frac{x\,dx}{1+x^2}$, by Simpson's rule.

We shall divide the range 0 to 8 into 8 equal intervals. Corresponding to the abscissae 0, 1, ... 8, the values of the ordinates are got from the equation $y = \frac{x}{1+x^2}$ and they are 0, $\frac{1}{2}$, $\frac{2}{5}$, $\frac{3}{10}$, $\frac{4}{17}$, $\frac{5}{26}$, $\frac{6}{37}$, $\frac{7}{50}$ and $\frac{8}{65}$.

By Simpson's rule, the area

$$= \frac{1}{2}\left[0 + \frac{8}{65} + 2\left(\frac{2}{5} + \frac{4}{17} + \frac{6}{37}\right) + 4\left(\frac{1}{2} + \frac{3}{10} + \frac{5}{26} + \frac{7}{50}\right)\right]$$

$$= \tfrac{1}{2}\left[\cdot 1231 + 1\cdot 5950 + 4\cdot 5292\right] = 2\cdot 0824.$$

By actual integration, $\int_0^8 \frac{x\,dx}{1+x^2} = \frac{1}{2}\Big[\log_e (1+x^2)\Big]_0^8$.

$$= \tfrac{1}{2}\log_e 65 = \tfrac{1}{2}\{\log_{10} 65 \times \log_e 10\} = 2\cdot 086.$$

Thus the error in using Simpson's rule is ·2 per cent.

Exercises LXXII.

1. Applying the trapezoidal rule and dividing the range into four equal intervals, show that $\int_0^1 \frac{dx}{1+x^2} = \cdot 7828$ approximately. Hence determine an approximate value of π. (B.Sc. Sub. 41)

2. Plot the function $4y + x^2 = 64$ for values of x between -8 and 8; then determine the area bounded by the curve and the x-axis (i) by Simpson's rule and (ii) by direct integration. (B.Sc. Sub. 45)

3. Find a so that the point (5, 10) may lie on the curve $y = ax^3$. Using Simpson's rule and taking ordinates corresponding to integral values of x, calculate the area between the curve, the x-axis and the lines $x = 1$, $x = 5$. By how much does this differ from the true area? (B.Sc. Sub. 46)

4. Evaluate $\int_1^{10} \frac{dx}{x}$ by the trapezoidal rule dividing the range

into nine equal intervals. Check this value with the result obtained by direct integration.

5. Compute $\int_0^5 \sqrt{125 - x^3}\, dx$ by the trapezoidal rule, dividing the range into five equal intervals.

6. Calculate $\int_0^{10} x^2\, dx$ by Simpson's rule, taking ten intervals.

7. Find an approximate value for $\log_e 2$ and thence for e from $\int_1^2 x^{-1}\, dx$.

8. Estimate, by Simpson's rule, the area between $xy = 12$, the axis of x, $x = 1$ and $x = 4$. Check the accuracy of the result by integration.

9. Calculate the approximate values of $\int_1^5 \frac{x\, dx}{\sqrt{36 - x^2}}$ both by the trapezoidal and Simpson's rules. Check their accuracy by actual integration. (The number of intervals may be taken as 4.)

10. Evaluate $\int_0^{\pi/2} x \cot x\, dx$, taking six strips. (B.Sc. 56 M)

§ 60. Volume of solid of revolution.

When a curve rotates about an axis, it generates the surface of a solid of revolution. The section of the solid at right angles to the axis is always a circle.

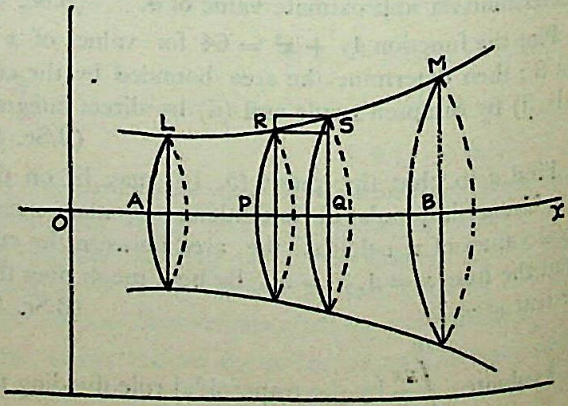

Fig. 63

We shall, for the sake of simplicity, suppose that the generating curve does not cut the axis of revolution.

The curve $y=f(x)$, the ordinates $x=a$, $x=b$ and the portion of the x-axis between them enclose a certain area.

To find a formula for the volume generated when the above area revolves completely about the x-axis.

Let the ordinates AL and BM be $x=a$ and $x=b$ respectively. Let R be any point (x, y) on the curve and S a neighbouring point $(x+\Delta x, y+\Delta y)$ on it. Draw the ordinates RP, SQ. When the curve rotates about the x-axis, every point on the curve describes a circle whose centre is the foot of the ordinate and radius the ordinate. Thus R describes a circle, centre P and radius $PR=y$. The circles described by L, R, S, M are indicated in the figure.

Let V be the volume of the solid of revolution between the circular sections through A and P. Then $V+\Delta V$ is the volume of the solid between the circular sections through A and Q so that ΔV represents the volume of the solid generated by the revolution of the arc RS. The distance between the two faces through P and Q is $PQ=\Delta x$ and the areas of the circles described by R and S are πy^2 and $\pi(y+\Delta y)^2$ respectively.

$\Delta V >$ the volume of the inner cylinder of height Δx, the base being the circle PR, and $<$ the volume of the outer cylinder of height Δx, the base being the circle QS.

i.e., $\Delta V > \pi y^2 \Delta x$ and $< \pi (y+\Delta y)^2 \Delta x$.

$\therefore\ \pi y^2 \Delta x < \Delta V < \pi (y+\Delta y)^2 \Delta x.$

Hence $\pi y^2 < \dfrac{\Delta V}{\Delta x} < \pi (y+\Delta y)^2$.

As $\Delta x \to 0$, $\dfrac{\Delta V}{\Delta x} \to \dfrac{dV}{dx}$ and $\dfrac{dV}{dx}$ lies between πy^2 and $\pi (y+\Delta y)^2$ which tends to πy^2 as Δy also $\to 0$.

$$\therefore\ \frac{dV}{dx} = \pi y^2.$$

$$V = \pi \int y^2\,dx + C = \pi \int \{f(x)\}^2\,dx + C.$$

Let $\pi \int \{f(x)\}^2\,dx = F(x)$.

Then $V = F(x) + C$. (1)

When $x=a$, $V=0$ (by definition of V).

$$\therefore\ 0 = F(a) + C.$$

into nine equal intervals. Check this value with the result obtained by direct integration.

5. Compute $\int_0^5 \sqrt{125 - x^3}\, dx$ by the trapezoidal rule, dividing the range into five equal intervals.

6. Calculate $\int_0^{10} x^3\, dx$ by Simpson's rule, taking ten intervals.

7. Find an approximate value for $\log_e 2$ and thence for e from $\int_1^2 x^{-1}\, dx$.

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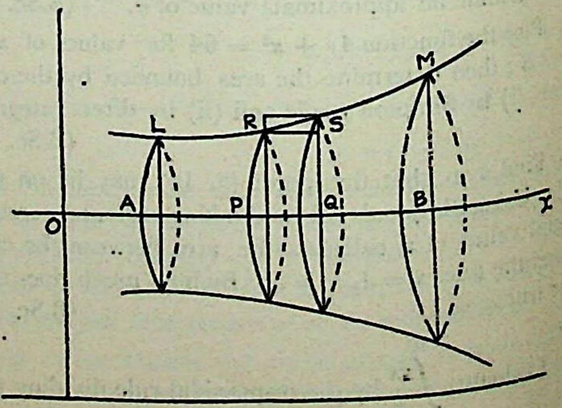

Fig. 63

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Let the ordinates AL and BM be $x = a$ and $x = b$ respectively. Let R be any point (x, y) on the curve and S a neighbouring point $(x + \Delta x, y + \Delta y)$ on it. Draw the ordinates RP, SQ. When the curve rotates about the x-axis, every point on the curve describes a circle whose centre is the foot of the ordinate and radius the ordinate. Thus R describes a circle, centre P and radius $PR = y$. The circles described by L, R, S, M are indicated in the figure.

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$\Delta V >$ the volume of the inner cylinder of height Δx, the base being the circle PR, and $<$ the volume of the outer cylinder of height Δx, the base being the circle QS.

i.e., $\Delta V > \pi y^2 \Delta x$ and $< \pi (y + \Delta y)^2 \Delta x$.

$\therefore \quad \pi y^2 \Delta x < \Delta V < \pi (y + \Delta y)^2 \Delta x.$

Hence $\pi y^2 < \dfrac{\Delta V}{\Delta x} < \pi (y + \Delta y)^2.$

As $\Delta x \to 0$, $\dfrac{\Delta V}{\Delta x} \to \dfrac{dV}{dx}$ and $\dfrac{dV}{dx}$ lies between πy^2 and $\pi (y + \Delta y)^2$ which tends to πy^2 as Δy also $\to 0$.

$$\therefore \quad \frac{dV}{dx} = \pi y^2.$$

$$V = \pi \int y^2\, dx + C = \pi \int \{ f(x) \}^2\, dx + C.$$

Let $\pi \int \{ f(x) \}^2\, dx = F(x)$.

Then $V = F(x) + C$.

When $x = a$, $V = 0$ (by definition of V).

$$\therefore \quad 0 = F(a) + C. \qquad (1)$$

When $x = b$, $V =$ the required volume $= F(b) + C.$ (2)

$\therefore$ The required volume $= F(b) - F(a)$ on subtracting (1) from (2)

$$= \Big[F(x) \Big]_a^b = \pi \int_a^b y^2\, dx.$$

Note.—The volume generated when the area bounded by a curve, the y-axis and abscissae $y = c$, $y = d$ revolves about the y-axis is $\int_c^d \pi x^2\, dy$.

Examples.

Ex. 1. A sphere of radius a is divided into two parts by a plane distant $a/2$ from the centre. Show that the ratio of the volumes of the two parts is 5 : 27.

When a semicircle of radius a revolves about its bounding diameter, a sphere of radius a is generated. The plane, at distance $a/2$ from the centre, divides the sphere into two spherical segments, major and minor.

The equation of the generating circle referred to its centre as origin is $x^2 + y^2 = a^2$.

Volume of the minor spherical segment $= \pi \int_{a/2}^{a} y^2\, dx$

$$= \pi \int_{a/2}^{a} (a^2 - x^2)\, dx = \pi \left(a^2 x - \frac{x^3}{3} \right)_{a/2}^{a} = \frac{5\pi a^3}{24}.$$

The volume of the major spherical segment $= \pi \int_{-a}^{a/2} y^2\, dx$

$$= \pi \int_{-a}^{a/2} (a^2 - x^2)\, dx = \pi \left[a^2 x - \frac{x^3}{3} \right]_{-a}^{a/2} = \frac{9\pi}{8} a^3.$$

Hence the ratio of the two volumes $= 5 : 27$.

Ex. 2. Find the volume when the loop of the curve $y^2 = x(2x - 1)^2$ revolves about the x-axis.

The curve crosses the x-axis when $x = 0$ and $x = \frac{1}{2}$.

Thus the limits for the loop are 0 and $\frac{1}{2}$.

The volume generated when the loop revolves about the x-axis is

$$\pi\int_0^{\frac{1}{2}} y^2\,dx = \pi\int_0^{\frac{1}{2}} x\,(2x-1)^2\,dx = \pi\int_0^{\frac{1}{2}} (4x^3-4x^2+x)\,dx$$

$$= \pi\left[x^4 - \frac{4x^3}{3} + \frac{x^2}{2}\right]_0^{\frac{1}{2}} = \frac{\pi}{48}.$$

Ex. 3. The area bounded by the curve $x = a\cos^3\theta$; $y = a\sin^3\theta$ and lying in the first quadrant revolves about the x-axis. Find the volume of the solid generated.

The curve is known as the four-cusped hypocycloid (*vide* page 234 for tracing of the curve).

The curve crosses the x-axis when $y=0$, *i.e.*, $\theta = 0$ and the y-axis when $x = 0$, *i.e.*, $\theta = \pi/2$. These are the limits of integration for θ in the case of the solid of revolution required.

The volume of the solid generated $= \pi\int y^2\,dx$

$$= -3\pi a^3 \int_{\pi/2}^{0} \sin^7\theta\cos^2\theta\,d\theta \text{ as } y = a\sin^3\theta \text{ and}$$

$$dx = -3a\cos^2\theta\sin\theta\,d\theta;$$

$$= +3\pi a^3\,\frac{6}{9}\,\frac{4}{7}\,\frac{2}{5}\,\frac{1}{3} = \frac{16\pi a^3}{105}.$$

Ex. 4. The part of a parabola cut off by the latus rectum is rotated (*i*) about the latus rectum (*ii*) about the axis. Show that the volumes generated are in the ratio 16 : 15. (B.Sc. 52 M)

Let the parabola be $y^2 = 4ax$.

(*i*) The equation of the latus rectum is $x = a$.

When the area of the parabola cut off by the latus rectum revolves about it, any point (x, y) on the parabola describes a circle of radius $a - x$.

Hence the volume generated by revolution about the latus rectum

$$= \pi\int_{-2a}^{+2a} (a-x)^2\,dy = \pi\int_{-2a}^{+2a}\left(a - \frac{y^2}{4a}\right)^2 dy$$

$$= \pi\int_{-2a}^{+2a}\left(a^2 - \frac{y^2}{2} + \frac{y^4}{16a^2}\right) dy = \pi\left[a^2y - \frac{y^3}{6} + \frac{y^5}{80a^2}\right]_{-2a}^{+2a} = \frac{32\pi a^3}{15}.$$

(*ii*) The volume generated by revolving the above area about the axis of the parabola, *i.e.*, $y = 0$ is

$$\pi \int_0^a y^2\, dx = \pi \cdot 4a \int_0^a x\, dx = 2\pi a \left(x^2\right)_0^a = 2\pi a^3.$$

Hence the required ratio is 16 : 15.

Exercises LXXIII.

1. Show that the volume of a spherical segment of height h, the radius of whose base is c is $\frac{\pi h}{6}(3c^2 + h^2)$.

2. A solid hemisphere of radius a is divided into two parts of equal volume by a plane parallel to the plane face at a distance x from it. Show that x satisfies the equation

$$x^3 - 3a^2x + a^3 = 0.$$

(B.Sc. '52 T.U. ; B.Sc. Sub. 42)

3. Find the volume generated by revolving the area of the parabola $y^2 = 4ax$ bounded by the ordinate $x = h$ about its axis. Verify that this volume is half the volume of the circumscribing cylinder.

4. The area included between the parabola $y^2 = 4ax$, the y-axis and the lines $y = \pm h$ is rotated about the y-axis. Find the volume thus generated. (B.A. 41 M)

5. (*i*) Compare the volumes generated when the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ revolves about the major and minor axes.

(*ii*) Find the volume of the spheroid generated by rotating the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ about the minor axis. (B.A. 47 S)

6. Find the volume of the solid generated when the area included between the curve $y^2 = x^3$ and (*i*) the straight line $x = h$, (*ii*) the curve $x^2 = y^3$ revolves about the x-axis.

(B.Sc. 52 T.U. ; B.Sc. 60)

7. Trace the curve $y^2 = x^2 \frac{(a + x)}{(b - x)}$ and find the volume of the solid obtained by rotating (*i*) the loop, (*ii*) the area between the origin and the asymptote about the x-axis.

(B.A. 36 M ; B.Sc. 52 T.U.)

8. The segments of the rectangular hyperbola $x^2 - y^2 = a^2$, cut off by $x = 2a$, revolves about the x-axis. Find the volume generated. (B.Sc. Sub. 38)

9. Find the volume generated by the revolution of an equilateral triangle of side a about one of its sides. (B.Sc. Sub. 45)

10. Find the volume of a right circular cone by integration.

11. Find the volume generated by the revolution of the loop of the curve (i) $y^2 = x^4(x+2)$. (B.A. 47 M)

(ii) $ay^2 = x^2(a-x)$ about the x-axis.

12. The figure bounded by the quadrant of a circle of radius a and the two tangents at its ends revolves about one of these tangents. Find the volume of the solid generated. (B.Sc. 51 M)

13. Find the volume generated by revolving the cycloid $x = a(\theta - \sin\theta)$; $y = a(1-\cos\theta)$ about its (i) base, (ii) tangent at the vertex.

14. The tractrix $x = a(\log\cot\theta/2 - \cos\theta)$; $y = a\sin\theta$ rotates about its asymptote, *viz.*, the x-axis. Find the volume formed.

15. Find the volume formed when the area bounded by the axis of x, the catenary $y = c\cosh\dfrac{x}{c}$ and the ordinates $x = \pm c$ rotates about the x-axis.

16. The segment of the parabola $y^2 = 4ax$ which is cut off by the latus rectum revolves about the directrix. Prove that the volume of the annular solid generated is $\dfrac{128}{15}\pi a^3$. (B.E. 50)

17. A sector of a circle of radius a and angle 60° rotates about its middle radius. Find the volume of the solid. (B.E. 51)

18. Find the area A enclosed by the x-axis and the curve $y = a(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x)$ between the ordinates at $0, \pi$; also the volume V obtained by rotating this area about the x-axis. Show that $4V = a\pi^2 A$. (B.Sc. Sub. 50)

19. If the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ revolves about the x-axis, show that the volume included between the surface thus generated, the cone generated by the asymptotes and the two planes perpendicular to the x-axis, at a distance h apart, is equal to that of a circular cylinder of height h and radius b.

20. The part of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cut off by the latus rectum revolves about the tangent at the vertex. Find the volume of the reel thus generated. (B.A. 36 M)

21. A segment of a circle of radius r cut off by a chord of length $2l$ is rotated about the chord. Find the volume of the spindle so formed. (B.Sc. 41 M)

22. Prove that the volume generated by rotating the loop of the curve $y^2(a+x)=x^2(a-x)$ about the x-axis is $2\pi a^3(\log 2-\frac{2}{3})$.

23. Find the area between the curve $(a-x)y^2=a^2x$ and its asymptote. If this curve revolves about its asymptote, find the volume of the solid generated. (B.E. 55 M)

24. Determine the volume generated by the revolution of the loop of the curve $x=t^2$, $y=t-\frac{t^3}{3}$ about the x-axis. (B.E. 55 S)

25. The curve $y^2(a+x)=x^2(3a-x)$ revolves about the x-axis. Prove that the volume of the solid generated by the loop of the curve is $\pi(8\log_e 2-3)a^3$. (B.Sc. 55 M)

26. The smaller part of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, cut off by the straight line $x=\frac{a}{2}$, is rotated about this line. Prove that the volume generated is $\pi a^2 b\left(\frac{3\sqrt{3}}{4}-\frac{\pi}{3}\right)$. (B.Sc. 66 M)

§ 61. Length of a Curve.

§ 61·1. Cartesian coordinates. We have already seen that the length of an arc of a curve $y=f(x)$ is given by the formula $\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$. The length of the curve between two points (x_1, y_1) and (x_2, y_2) on it is

$$\int_{x_1}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Examples.

Ex. 1. Find the length of the arc of the parabola $y^2=4ax$ from the vertex to any point (x, y) on the curve

$$ds=\sqrt{dy^2+dx^2}.$$

Differentiating, $y^2=4ax$, $y\,dy=2a\,dx$.

$$\therefore\ ds=\sqrt{dy^2+\frac{y^2(dy)^2}{4a^2}}=\frac{dy}{2a}\sqrt{4a^2+y^2}.$$

$$\therefore \quad s = \frac{1}{2a}\int_0^y \sqrt{4a^2 + y^2}\, dy$$

$$= \frac{1}{4a}\left[y\sqrt{4a^2+y^2} + 4a^2 \sinh^{-1}\frac{y}{2a} \right]_0^y$$

$$= \frac{1}{4a}\left[y\sqrt{4a^2+y^2} + 4a^2 \sinh^{-1}\frac{y}{2a} \right].$$

(It is to be noted that x has not been taken as the independent variable for s, in that case, will be $= \int_0^x \sqrt{\frac{a+x}{x}}\, dx$. The integrand has an infinity at the lower limit and the discussion of the convergence of this integral is, strictly speaking, beyond the scope of the book.)

Ex. 2. Find the length of the complete arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

$dx = a(1 - \cos\theta)\, d\theta$ and $dy = a \sin\theta\, d\theta$.

Hence $ds = \sqrt{dx^2 + dy^2} = a\, d\theta\, \sqrt{(1-\cos\theta)^2 + \sin^2\theta}$

$$= a \,.\, 2 \sin\frac{\theta}{2}\, d\theta.$$

$\therefore$ Length of the complete arch

$$= \int_0^{2\pi} 2a \sin\theta/2 \,.\, d\theta = 4a\left(-\cos\frac{\theta}{2}\right)_0^{2\pi} = 8a.$$

Ex. 3. Find the length of one loop of the curve $3ay^2 = x(x-a)^2$. (B.E. 50)

We note that the curve is symmetrical about the x-axis and the limits for the loop are 0 and a. The length of the arc of the loop = twice that of the arc of the loop about the x-axis, by symmetry.

Differentiating the equation of the curve,

$6ay\, dy/dx = (x-a)^2 + 2x(x-a) = (x-a)(3x-a)$.

The arc of the loop above the x-axis

$$= \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$=\int_0^a \sqrt{1+\frac{(x-a)^2(3x-a)^2}{36a^2y^2}}\,dx$$

$$=\int_0^a \sqrt{1+\frac{(3x-a)^2}{12ax}}\,dx \text{ on substituting } x(x-a)^2 \text{ for } 3ay^2$$

$$=\frac{1}{2\sqrt{3a}}\int_0^a \sqrt{\frac{12ax+(3x-a)^2}{x}}\,dx$$

$$=\frac{1}{2\sqrt{3a}}\int_0^a \frac{(3x+a)}{\sqrt{x}}\,dx=\frac{1}{2\sqrt{3a}}\int_0^a\left(3\sqrt{x}+\frac{a}{\sqrt{x}}\right)dx$$

$$=\frac{1}{2\sqrt{3a}}\Big[2x^{3/2}+2a\sqrt{x}\Big]_0^a=\frac{2a}{\sqrt{3}}.$$

$\therefore$ The entire length of the loop $=4a/\sqrt{3}$.

Exercises LXXIV.

1. Find the length of an arc of a circle by integration.

2. Prove that the length of the arc of the parabola $y^2=4ax$ cut off by the latus rectum is

$$2a[\sqrt{2}+\log(1+\sqrt{2})].$$ (B.Sc. 51 T.U.)

3. Find the length of the curve $y=\log\sec x$ between the points given by $x=0$ and $x=\pi/3$.

(B.Sc. 53 T.U. ; B.Sc. Sub. 39)

4. (*i*) Find the length of the arc of the catenary $y=c\cosh\frac{x}{c}$ measured from its vertex to the point (x, y). (B.Sc. 53 Os.U.)

(*ii*) Show that $y^2=c^2+s^2$. (B.Sc. 52 M.)

(*iii*) Show that the area between the catenary, the x-axis and the ordinates of two points on it varies as the length of the intervening arc.

(*iv*) If s_1 is the length of the arc $y=\cosh x$ which lies between $x=a$ and $x=5a$ and s_2 is the length of the same curve between $x=3a$ and $x=7a$, prove that $s_1\cosh 5a=s_2\cosh 3a$.

(B.Sc. 39 M.)

5. Find the length of the curve $y=\log x$ between the points whose abscissae are 1 and a $(a>1)$. (B.Sc. Sub. 44)

6. (*a*) Show that the length of the arc of a curve $27y^2 = 4x^3$ measured between the origin and the point $(6, 4\sqrt{2})$ is equal to $2(3\sqrt{3} - 1)$.
(B.Sc. 49 M)

(*b*) Show that the length of the curve $x^3 = y^2$ from $x = 0$ to $x = 1$ is $\frac{1}{27}(13\sqrt{13} - 8)$.
(B.A. Sub. 56)

7. Find the length of the boundary of the loop of the curve $9ay^2 = x(x - 3a)^2$.
(B.Sc. 50 M)

8. Show that the length of the arc of the parabola $y^2 = 4ax$ which is intercepted between the points of intersection of the parabola and $3y = 8x$ is $a(\log 2 + 15/16)$.
(B.Sc. 51 M)

9. Find the length of the curve $9x^2 = 4(1 + y^2)^3$ from the point $\left(\frac{2}{3}, 0\right)$ to the point $\left(\frac{10\sqrt{5}}{3}, 2\right)$.
(B.Sc. 52 M)

10. Find the length of the portion of the curve

(*i*) $3y = \sqrt{x}(x - 3)$ between $x = 0$ and $y = 9$.
(B.Sc. Sub. 48)

(*ii*) $y = \log \coth \frac{x}{2}$ between the points (x_1, y_1) and (x_2, y_2).
(B.Sc. Sub. 51)

11. (*i*) Show that the entire length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $6a$.

(*ii*) Show that the whole length of $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$ is $4 \cdot \left(\frac{a^2}{b} + \frac{b^2}{a}\right)$.
(B.E. 51)

12. If ρ be the radius of curvature at the point $\theta = 45°$ of the curve $x = a \log \sec \theta$, $y = a(\tan \theta - \theta)$ and s the length of the curve measured from the origin to the same point, show that $\rho = s + a$.
(B.Sc. 52 Os.U.)

13. A curve is given by the equations

$$x = a(\cos \theta + \theta \sin \theta)\ ;\ y = a(\sin \theta - \theta \cos \theta).$$

Find the length of the arc from $\theta = 0$ to $\theta = a$.

14. Find the length of the arc of

(*i*) the curve $4y^2 = (x + 2)^3$ from $x = 2$ to $x = 7$.
(B.E. 53)

(*ii*) the curve $y = \log \frac{e^x - 1}{e^x + 1}$ from $x = 1$ to $x = 2$.

(*iii*) $6xy = 3 + y^4$ between the points whose ordinates are 1 and 4. (B.E. 52)

$$\left[\textit{Hint.}\ \text{Use } ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy.\right]$$

15. Trace the curve $x = 2\theta - \sin 2\theta$; $y = 2 \sin^2 \theta$ as θ varies from 0 to π and find its length. (B.A. 46 M)

§ 61·2. Polar coordinates.

In polar coordinates, the element of arc is given by

$\dfrac{ds}{d\theta} = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}\, d\theta$ (*vide* § 39·6). The equation of the curve $r = f(\theta)$ being known, $\dfrac{dr}{d\theta}$ can be found in terms of θ. Hence the length of the arc between the points whose vectorial angles are θ_1 and θ_2 is

$$\int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Example. Prove that the perimeter of the cardioid $r = a(1 + \cos\theta)$ is $8a$. (B.A. 51 M)

Since the curve is symmetrical about the initial line the perimeter is double the length of the curve above the initial line.

Differentiating the equation of the curve, $dr/d\theta = -a \sin\theta$.

Hence the perimeter $2\displaystyle\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$$= 2\int_0^{\pi} \sqrt{a^2(1+\cos\theta)^2 + a^2 \sin^2\theta}\, d\theta$$

$$= 2a\int_0^{\pi} \sqrt{2(1+\cos\theta)}\, d\theta = 4a\int_0^{\pi} \cos\theta/2\, d\theta$$

$$= 4a\left[2 \sin\frac{\theta}{2}\right]_0^{\pi} = 8a.$$

Exercises LXXV.

1. Find the length of the arc of the parabola $r(1 + \cos\theta) = 2a$ cut off by the latus rectum. (B.Sc. Sub. 50)

2. Find the length of (*i*) the spiral of Archimedes $r = a\theta$, from $\theta = 0$ to $\theta = \theta_1$, (*ii*) the hyperbolic spiral $r\theta = a$ from (r_1, θ_1) to (r_2, θ_2).

3. Prove that the length of the equiangular spiral $r = ae^{\theta \cot a}$ between two radii of lengths r_1 and r_2 is $(r_1 \sim r_2) \sec a$.
(B.Sc. 51 T.U.)

4. Find the length of the curve $r = a \cos^3 \theta/3$.

5. Find the length of the arc of the cissoid
$$r = 2a \tan \theta \sin \theta \text{ from } \theta = 0 \text{ to } \theta = \pi/4.$$

6. Show that the upper half of the cardioid $r = a (1 + \cos \theta)$ is bisected by the line $\theta = \pi/3$.

§ 62. Area of a surface of revolution.

A surface of revolution is generated by revolving the arc LM of the curve $y = f(x)$ about the x-axis.

(*Vide* Figure 63, Page 332)

Let the arcual distance of R (x, y) on the curve from a fixed point on the curve be s and let arc RS be Δs. Let S be the area of surface generated by revolving arc LR about the x-axis and $S + \Delta S$ be the area obtained by revolving the arc LS about the x-axis; so ΔS is the elementary area of the surface of the solid generated by revolving the arc RS about the x-axis. This elementary area is almost in the form of frustum of a cone, the end sections being the circular sections through R and S of radii y and $y + \Delta y$.

As the lateral area of the frustum of a cone of a revolution is equal to the circumference of the middle section multiplied by the slant length,

$$\Delta S = \tfrac{1}{2} \{ 2\pi y + 2\pi (y + \Delta y) \} RS \text{ approximately}$$

$$= \pi \{ 2y + \Delta y \} \Delta s \text{ as chord } RS \text{ is approximately equal to arc } RS$$

$$= 2\pi y \, \Delta s + \pi \, \Delta y \, . \, \Delta s.$$

$$\therefore \frac{\Delta S}{\Delta s} = 2\pi y + \pi \frac{\Delta y}{\Delta s} \Delta s.$$

In the limit when $\Delta s \to 0$, $\dfrac{dS}{ds} = 2\pi y$.

$\therefore$ $S = 2\pi \int y \, ds$ between suitable limits.

In cartesian coordinates, $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$; the area of the surface of the solid generated by revolving arc LM about x-axis is

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

In polar coordinates, $ds = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta.$

$$S = 2\pi \int r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$ between suitable limits for θ.

Note.—If the arc rotates about the axis of y, the area of the surface of the solid formed is obtained by writing $2\pi x$ for $2\pi y$ in the above formulae.

Examples.

Ex. 1. Find the area of a belt of a sphere of radius a intercepted between two parallel planes at distances b and c from the centre on the same side.

A sphere is generated by revolving a semi-circle about its bounding diameter. The equation of the generating circle is $x^2 + y^2 = a^2$. Let P be any point (x, y) on the circle whose arcual distance from A on the circle is s. If $\angle AOP$ be θ, $s = a\theta$.

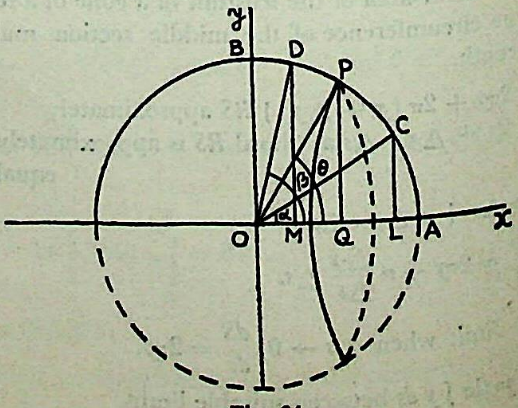

Fig. 64

Let $OL = b$ and $OM = c$.

The arc CD of the circle revolves about OA and generates the belt.

Let $\angle AOC$ be α and $\angle AOD = \beta$.

Area of the surface generated

$$= 2\pi \int y\, ds = 2\pi \int_{\alpha}^{\beta} a \sin\theta \,.\, a\, d\theta, \text{ as } ds = a\, d\theta$$

$$= 2\pi a^2 \Big[-\cos\theta\Big]_{\alpha}^{\beta} = 2\pi a^2 (\cos\alpha - \cos\beta)$$

$$= 2\pi a\,(OL - OM) \text{ as } OL = a\cos\alpha \text{ and } OM = a\cos\beta$$

$$= 2\pi a\,(b - c).$$

Cor. The curved surface of a hemisphere is got by putting $b = a$ and $c = 0$ in the above result. Hence the area of the surface of a hemisphere $= 2\pi a^2$.

Ex. 2. Find the area of the surface generated by revolving the arc of the catenary $y = c\cosh\frac{x}{c}$ from $x = 0$ to $x = c$ about x-axis.

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

As $y = c\cosh\frac{x}{c}$, $\frac{dy}{dx} = \sinh\frac{x}{c}$.

$$\therefore ds = \sqrt{1 + \sinh^2\frac{x}{c}}\, dx = \cosh\frac{x}{c}\, dx.$$

Hence, the area of the surface required

$$= 2\pi \int y\, ds = 2\pi c \int_0^c \cosh^2\frac{x}{c}\, dx$$

$$= \pi c \int_0^c \left(1 + \cosh\frac{2x}{c}\right) dx = \pi c \left[x + \frac{c}{2}\sinh\frac{2x}{c}\right]_0^c$$

$$= \frac{\pi c^2}{2}\,[2 + \sinh 2].$$

Ex. 3. Find the surface of the solid generated by the revolution about the x-axis of the loop of the curve $x = t^2$; $y = t - t^3/3$. (B.Sc. 44 M ; B.E. 52)

The curve crosses the x-axis at the points given by $y=0$, *i.e.*, $t=0$ and $t=\sqrt{3}$. These are the limits for the loop.

[This is otherwise evident ; by eliminating t, the equation of the curve is $9y^2 = x(3-x)^2$.]

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

As $x = t^2, \frac{dx}{dt} = 2t$; $y = t - \frac{t^3}{3}, \frac{dy}{dt} = 1 - t^2$.

$\therefore\ ds = \sqrt{4t^2 + (1-t^2)^2}\, dt = (1+t^2)\, dt.$

The area of the surface required

$$= 2\pi \int y\, ds = 2\pi \int_0^{\sqrt{3}} \left(t - \frac{t^3}{3}\right)(1+t^2)\, dt$$

$$= 2\pi \int_0^{\sqrt{3}} \left(t + \frac{2t^3}{3} - \frac{t^5}{3}\right) dt = 2\pi \left[\frac{t^2}{2} + \frac{t^4}{6} - \frac{t^6}{18}\right]_0^{\sqrt{3}}$$

$$= \pi(3+3-3) = 3\pi.$$

Ex. 4. Find the area of the surface of the solid generated by rotating the cardioid $r = a(1+\cos\theta)$ about its line of symmetry.

The initial line cuts the curve at $\theta = 0$ and $\theta = \pi$.

Differentiating $r = a(1+\cos\theta)$, $\frac{dr}{d\theta} = -a\sin\theta$.

$$ds = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta = \sqrt{a^2\sin^2\theta + a^2(1+\cos\theta)^2}\, d\theta$$

$$= a\sqrt{2+2\cos\theta}\, d\theta = 2a\cos\theta/2\, d\theta.$$

The surface area $S = 2\pi \int y ds = 2\pi \int_0^{\pi} r\sin\theta \cdot 2a\cos\theta/2\, d\theta$

$$= 4\pi a^2 \int_0^{\pi} (1+\cos\theta)\sin\theta\cos\theta/2\, d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \cos^4\theta/2 \sin\theta/2\, d\theta$$

$$= 16\pi a^2 \left[-\frac{2}{5}\cos^5\theta/2\right]_0^{\pi} = \frac{32\pi a^2}{5}.$$

Exercises LXXVI.

1. Find the ratio in which the surface area of a sphere is divided by a plane distant x from its centre.

2. Find the area of the curved surface obtained by revolving about the x-axis

(*i*) the arc of the parabola $y^2 = 4ax$ between the points whose abscissae are 0 and x.

(*ii*) the arc of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(*iii*) the curve $6xy = x^4 + 3$ from $x = 1$ to $x = 4$. (B.Sc. 51 T.U.)

(*iv*) the loop of the curve $8a^2y^2 = a^2x^2 - x^4$.

(*v*) the loop of the curve $3ay^2 = x(x-a)^2$.

3. The arc of a quadrant of a circle rotates about (*i*) a tangent at one end of it (*ii*) its chord (*iii*) the tangent at its middle point. Find in each case the area of the surface generated.

4. A circle of radius r rotates about an axis in its own plane at a distance c ($> r$) from its centre. Find the volume and surface area of the surface generated. (B.Sc. Sub. 50)

(The above solid is called an *anchor ring*.)

5. Find the surface of the solid generated by the revolution of the cycloid $x = a(\theta - \sin\theta)$; $y = a(1 - \cos\theta)$ (*i*) about its base (*ii*) about the tangent at its vertex. (B.A. 51 M)

6. Find the area of the surface obtained by revolving $x = a\cos^3\theta$; $y = a\sin^3\theta$ about the x-axis. (B.Sc. 61 M)

7. Find the area of the surface formed by rotating $y^2 = x^3$ from $x = 0$ to $x = 4$ about the y-axis.

8. The arc of the parabola $y^2 = 4ax$ cut off by the latus rectum rotates about the tangent at the vertex ; find the area of the surface described.

9. The portion of the parabola $r(1 + \cos\theta) = 2a$ cut off by the latus rectum revolves about the axis. Find the surface area of the solid formed. (B.E. 46)

10. The curve $r = 4 + 2\cos\theta$ rotates about its axis. Find the surface area of the solid formed. (B.E. 49)

11. The curve $r = e^{\theta}$ from $\theta = 0$ to $\theta = \pi$ rotates about the initial line. Find the area of the surface of the solid generated.

12. The curve $r^2 = a^2 \cos 2\theta$ revolves about the initial line. Find the area of the surface generated. (B.Sc. Sub.)

13. A circular arc revolves about its chord. Prove that the area of the surface generated is $4\pi a^2 (\sin \alpha - \cos \alpha)$, where a is the radius and 2α the angle subtended by the arc at the centre.

14. Find the curved surface area of the zone of a sphere of radius a cut off by two parallel planes at distances a_1 and a_2 from one end of a diameter. (B.Sc. Comp. 59)

CHAPTER XIV

PHYSICAL APPLICATIONS OF INTEGRATION

§ 63·1. **Centroid.** Let a system of particles of masses $m_1, m_2, \ldots\ldots$ be situated at points in a plane whose coordinates are $(x_1, y_1), (x_2, y_2), \ldots\ldots$ with reference to fixed axes. The point whose coordinates are $(\bar{x}, \bar{y})$ given by the equation $\bar{x} = \dfrac{\Sigma mx}{\Sigma m}$, $\bar{y} = \dfrac{\Sigma my}{\Sigma m}$ is called the *centre of mass* of the system.

If, instead of a finite number of particles, we have a continuous distribution of matter, as in the case of a lamina or a rigid body, the summations in the above formulae become definite integrals. Thus the centre of mass of a body of mass M is given by the equations $M\bar{x} = \int x\, dm$ and $M\bar{y} = \int y\, dm$, where dm is an element of mass of the body concentrated at the point (x, y). If the lamina or the body be of uniform density, the centre of mass is known as the *centroid*.

§ 63·2. **Centre of mass of an arc.**

Let P be any point (x, y) of a plane arc, whose arcual distance from a fixed point on the curve is s, and ρ be the line density (*i.e.*, mass per unit length of the curve) at P and let PQ be an elementary arc ds. Then the elementary mass dm can be taken as $\rho\, ds$ and supposed to act at $P(x, y)$. The formulae for the centre of mass of the arc are

$\bar{x} = \dfrac{\int \rho\, ds\, x}{\int \rho\, ds}$ and $\bar{y} = \dfrac{\int \rho\, ds\, y}{\int \rho\, ds}$ between suitable limits. ds is given by the formula $ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\, dx$ in cartesian coordinates and $ds = \sqrt{\left(\dfrac{dr}{d\theta}\right)^2 + r^2}\, d\theta$ in polar coordinates. When the equation of the curve is given, $\dfrac{dy}{dx}$ or $\dfrac{dr}{d\theta}$ can be calculated as the case may be. If the arc be uniform, ρ is a constant and the centroid of the arc is given by $\bar{x} = \dfrac{\int x\, ds}{\int ds}$ and $\bar{y} = \dfrac{\int y\, ds}{\int ds}$.

Examples.

Find the centroid of the arc of the parabola $y^2 = 4ax$ between the vertex and the point (x, y).

Here $y\dfrac{dy}{dx} = 2a.$

$$\therefore\ ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \frac{4a^2}{y^2}}\, dx$$
$$= \sqrt{\frac{x+a}{x}}\, dx.$$

$$\text{Hence } \bar{x} = \frac{\int_0^x x\, ds}{\int_a^x ds} = \frac{\int_0^x \sqrt{x(x+a)}\, dx}{s}$$

$$\text{and } \bar{y} = \frac{\int_0^x y\, ds}{\int_0^x ds} = 2\sqrt{a}\,\frac{\int_0^x \sqrt{x+a}\, dx}{s}.$$

(*i*) The denominator s

$$= \frac{1}{4a}\left[y\sqrt{4a^2 + y^2} + 4a^2 \sinh^{-1}\frac{y}{2a}\right]$$

by § 61·1, Ex. 1, page 339.

$$= \sqrt{x(x+a)} + a \log\frac{(\sqrt{x} + \sqrt{x+a})}{\sqrt{a}}$$

on putting $y^2 = 4ax$.

$$\text{(ii)} \int_0^x \sqrt{x(x+a)}\, dx = \int_0^x \sqrt{\left(x + \frac{a}{2}\right)^2 - \frac{a^2}{4}}\, dx$$

$$= \frac{a^2}{4}\int_0^\theta \sinh^2\theta\, d\theta \text{ on putting } x + \frac{a}{2} = \frac{a}{2}\cosh\theta;$$

$$dx = \frac{a}{2}\sinh\theta\, d\theta.$$

$$= \frac{a^2}{8}\int_0^\theta (\cosh 2\theta - 1)\, d\theta = \frac{a^2}{8}\left(\frac{\sinh 2\theta}{2} - \theta\right).$$

$$= \frac{a^2}{8}(\sinh\theta\cosh\theta - \theta)$$

$$=\frac{1}{2}\left[\left(x+\frac{a}{2}\right)\sqrt{\left(x+\frac{a}{2}\right)^2-\frac{a^2}{4}}-\frac{a^2}{4}\cosh^{-1}\frac{x+\frac{a}{2}}{\frac{a}{2}}\right].$$

$$=\frac{1}{2}\left(x+\frac{a}{2}\right)\sqrt{x^2+ax}-\frac{a^2}{8}\log\left(\frac{x+\frac{a}{2}+\sqrt{x^2+ax}}{\frac{a}{2}}\right).$$

(iii) $\int_0^x \sqrt{x+a}\,dx=\frac{2}{3}[(x+a)^{3/2}-a^{3/2}].$

Substituting these values of the integrals,

$\bar{x}$ and $\bar{y}$ are got.

§ 63·3. Centre of mass of a plane area.

Let δA be an element of area surrounding or at the point (x, y) and ρ be the density at the point. The element of mass $dm=\rho dA$ and the formulae for the centre of mass take the form

$$\bar{x}=\frac{\int \rho dAx}{\int \rho dA} \text{ and } \bar{y}=\frac{\int \rho dAy}{\int \rho dA}. \qquad (1)$$

If the area be of uniform density, it can be dissected into rectangular strips as in § 58·1, page 316.

Let the ordinate at distance x cut the curve in points whose ordinates are y_1 and y_2. Now dA can be taken as $(y_1-y_2)\,dx$, whose centre of mass has coordinates x and $\frac{y_1+y_2}{2}$ in the limit when $\delta x\to 0$. Hence

$$\bar{x}=\frac{\int x(y_1-y_2)\,dx}{\int (y_1-y_2)\,dx} \text{ and } \bar{y}=\frac{\int (y_1-y_2)\frac{(y_1+y_2)}{2}dx}{\int (y_1-y_2)\,dx}. \qquad (2)$$

From the equation of the curve, the values of y_1 and y_2 are known and the limits for x are such as to cover the area in question.

If the area be bounded by the arc $y=f(x)$, the ordinates $x=a$ and $x=b$ and the portion of the x-axis as in § 58·1, then $dA=y\,dx$ and this acts at $\left(x, \frac{y}{2}\right)$. Hence the centroid is given by

$$\bar{x}=\frac{\int_a^b xy\,dx}{\int_a^b y\,dx} \text{ and } \bar{y}=\frac{1}{2}\frac{\int_a^b y^2\,dx}{\int_a^b y\,dx}. \qquad (3)$$

If polar coordinates be employed and the area be uniform, dA can be taken to be almost a triangle and equal to $\frac{1}{2}r^2 d\theta$, (*vide* § 58·4, page 325). Its centre of mass is the median point of this triangle practically, *i.e.*, the point whose polar coordinates are $(\frac{2}{3}r, \theta)$ in the limit when $\delta\theta \to 0$.

Hence the centroid of the area is given by

$$\bar{x} = \frac{\int \frac{1}{2} r^2 d\theta \, \frac{2}{3} r \cos\theta}{\int \frac{1}{2} r^2 d\theta} = \frac{2}{3} \frac{\int r^3 \cos\theta \, d\theta}{\int r^2 d\theta},$$

$$\text{and } \bar{y} = \frac{\int \frac{1}{2} r^2 d\theta \, \frac{2}{3} r \sin\theta}{\int \frac{1}{2} r^2 d\theta} = \frac{2}{3} \frac{\int r^3 \sin\theta \, d\theta}{\int r^2 d\theta} \qquad (4)$$

where the equation of the curve $r = f(\theta)$ is known and the limits for θ to cover the area required are determined in any problem.

Examples.

Ex. 1. Find the centroid of an elliptic quadrant.

The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Using the formulae (3) above,

$$\bar{x} = \frac{\int_0^a xy \, dx}{\int_0^a y \, dx} \text{ as } x \text{ varies from } 0 \text{ to } a \text{ in the elliptic quadrant}$$

$$= \frac{\int_0^a x\, b \sqrt{1 - \frac{x^2}{a^2}} \, dx}{\int_0^a b \sqrt{1 - \frac{x^2}{a^2}} \, dx}.$$

Putting $x = a \sin\theta$ and $dx = a \cos\theta \, d\theta$ in both the numerator and denominator,

$$\bar{x} = a \frac{\int_0^{\pi/2} \sin\theta \cos^2\theta \, d\theta}{\int_0^{\pi/2} \cos^2\theta \, d\theta} = \frac{a\left[\frac{1}{3}\right]}{\frac{\pi}{4}} = \frac{4a}{3\pi}.$$

By symmetry, $\bar{y} = \frac{4b}{3\pi}$.

Cor. The centroid of a circular quadrant is $\left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right)$.

Ex. 2. Find the centroid of the arc and sector of a circle.

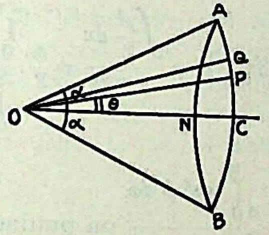

Fig. 65

Let the arc AB of a circle subtend an angle 2α radians at its centre O and let OC be the radius bisecting the arc.

By symmetry, the centre of the arc lies on OC.

Taking OC as the x-axis, let P be any point (x, y) on the arc and let COP be θ and arc CP be s.

Then $x = a\cos\theta$ and $s = a\theta$.

Hence, for the arc, $\bar{x} = \dfrac{\int x\,ds}{\int ds} = \dfrac{\int_{-\alpha}^{+\alpha} a\cos\theta\, a\,d\theta}{\int_{-\alpha}^{+\alpha} a\,d\theta} = a\dfrac{\sin\alpha}{\alpha}$.

For the sector AOB, the centroid lies on OC by symmetry. The element of area is $\frac{1}{2}a^2\,d\theta$ and hence, by formula (4) of § 63·3,

$$\bar{x} = \frac{\int_{-\alpha}^{+\alpha} \frac{1}{2}a^2\,d\theta\,\frac{2}{3}a\cos\theta}{\int_{-\alpha}^{+\alpha}\frac{1}{2}a^2\,d\theta} = \frac{2}{3}a\,\frac{\sin\alpha}{\alpha}.$$

Cor. Putting $\alpha = \pi/2$, the centroids of a semicircular arc and area lie on the middle radius at distances $\frac{2a}{\pi}$ and $\frac{4a}{3\pi}$ from the centre.

Ex. 3. Find the centroid of the area enclosed by the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ and the x-axis from cusp to cusp.

(B.Sc. Sub. 45)

The cusps correspond to $\theta = 0$ and $\theta = 2\pi$. (*Vide* Page 230.)

As the cycloid is symmetrical about the line $x=\pi$, the centroid lies on the line of symmetry. Hence $\bar{x}=\pi$ and $\bar{y}$ need only be evaluated.

From (3) of § 63·3, $$\bar{y}=\frac{\int \frac{y^2}{2}\,dx}{\int y\,dx}=\frac{a}{2}\cdot\frac{\int_0^{2\pi}(1-\cos\theta)^3\,d\theta}{\int_0^{2\pi}(1-\cos\theta)^2\,d\theta}$$

$$=\frac{a}{2}\frac{\int_0^{2\pi}8\cos^6\theta/2\,d\theta}{\int_0^{2\pi}4\cos^4\theta/2\,d\theta}=\frac{a}{2}\frac{\int_0^{\pi}\cos^6\phi\,d\phi}{\int_0^{\pi}\cos^4\phi\,d\phi}\text{ on putting }\theta/2=\phi$$

$$=a\frac{2\int_0^{\pi/2}\cos^6\phi\,d\phi}{2\int_0^{\pi/2}\cos^4\phi\,d\phi}=a\frac{\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}}{\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}}=\frac{5a}{6}.$$

Ex. 4. Find the centroid of one loop of the lemniscate $r^2=a^2\cos 2\theta$.

For the tracing of the curve, *vide* page 229.

By symmetry, the centroid lies on the initial line.

Hence $\bar{y}=0$ and $\bar{x}$ need only be calculated.

From (4) of § 63·3, $$\bar{x}=\frac{2}{3}\frac{\int_{-\pi/4}^{\pi/4}r^3\cos\theta\,d\theta}{\int_{-\pi/4}^{\pi/4}r^2\,d\theta}$$ as θ varies from $-\pi/4$ to $\pi/4$ in a loop;

$$=\frac{2a}{3}\frac{2\int_0^{\pi/4}(\cos 2\theta)^{3/2}\cos\theta\,d\theta}{2\int_0^{\pi/4}\cos 2\theta\,d\theta}=\frac{2a}{3}\frac{\int_0^{\pi/4}(1-2\sin^2\theta)^{3/2}\cos\theta\,d\theta}{\left\{\frac{\sin 2\theta}{2}\right\}_0^{\pi/4}}$$

$$=\frac{2\sqrt{2}a}{3}\int_0^{\pi/2}\cos^4\phi\,d\phi$$ on putting $\sqrt{2}\sin\theta=\sin\phi$ in the numerator;

$$=\frac{2\sqrt{2}a}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi a\sqrt{2}}{8}.$$

§ 63·4. Centroid of a solid of revolution.

Let the arc LM of the curve $y=f(x)$ (*vide* Fig. 63 of § 60, page 332) revolve about x-axis and generate a solid of revolution. The element of area $RSQP$ between the ordinates at distances x and $x+\Delta x$ from O generates the element of volume which is almost in the form of a cylinder. Hence the element of volume is $\pi y^2 \,\Delta x$ and its centroid lies on Ox at a distance x from O, when $\Delta x \to 0$. Taking the solid to be of uniform density, δm is proportional to $\pi y^2 \,\delta x$.

$$\therefore \quad \bar{x} = \frac{\int_a^b \pi y^2 \, dx \, x}{\int_a^b \pi y^2 \, dx} = \frac{\int_a^b y^2 x \, dx}{\int_a^b y^2 \, dx} \text{ and } \bar{y}=0.$$

Example. Find the centroid of a uniform solid hemisphere.

A hemisphere is generated by revolving the quadrant of the circle $x^2+y^2=a^2$ about one of its bounding radii taken to be the x-axis.

$$\therefore \quad \bar{x} = \frac{\int_0^a y^2 x \, dx}{\int_0^a y^2 \, dx} = \frac{\int_0^a (a^2-x^2)\, x \, dx}{\int_0^a (a^2-x^2) \, dx} \text{ as } y^2=a^2-x^2;$$

$$= \frac{\left(a^2 \frac{x^2}{2} - \frac{x^4}{4}\right)_0^a}{\left(a^2 x - \frac{x^3}{3}\right)_0^a} = \frac{3a}{8}. \quad \text{Obviously } \bar{y}=0.$$

§ 63·5. Centroid of surface of revolution.

By revolving arc LM (*vide* Fig. 63, page 332) about the x-axis, we get a surface of revolution. We saw in § 62 that the elementary area dS generated by the revolution of the arc RS about the x-axis is $2\pi y \, ds$ and this is almost a frustum of a cone. Its centroid lies on the x-axis at distance x from the origin, when $\Delta x \to 0$. Taking the surface to be of uniform density, δm is proportional to $2\pi y \, ds$.

$$\text{Hence, } \bar{x} = \frac{\int 2\pi y \, ds \,.\, x}{\int 2\pi y \, ds} = \frac{\int y x \, ds}{\int y \, ds} \text{ and } \bar{y}=0.$$

The limits are to be fixed suitably in each case by drawing the figure of the generating curve.

The formula

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \text{ or } ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

is used according as cartesian or polar co-ordinates are employed.

Examples.

Ex. 1. Find the centroid of a hollow hemisphere.

A hollow hemisphere is generated by revolving the arc AB of a quadrant of a circle about Ox. (*Vide* Fig. 64 of Ex. 1, page 344.)

Arc $AP = s = a\theta$ and $y = a \sin\theta$; $x = a\cos\theta$ and from A to B, θ varies from 0 to $\pi/2$.

$$\therefore \quad \bar{x} = \frac{\int y\,ds\,.\,x}{\int y\,ds} = a\,.\,\frac{\int_0^{\pi/2} \cos\theta \sin\theta\,d\theta}{\int_0^{\pi/2} \sin\theta\,d\theta}$$

$$= a\frac{\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}}{\left[-\cos\theta\right]_0^{\pi/2}} = \frac{a}{2}.$$

Evidently $\bar{y} = 0$.

Ex. 2. Find the centroid of the surface generated by revolving the cardioid $r = a(1 + \cos\theta)$ about the initial line $ds = 2a\cos(\theta/2)\,d\theta$ (*vide* Ex. 4, page 346).

$$\bar{x} = \frac{\int yx\,ds}{\int y\,ds} = \frac{\int_0^{\pi} r^2 \sin\theta\cos\theta\cos\theta/2.d\theta}{\int_0^{\pi} r\sin\theta\cos\theta/2.d\theta}$$

as $x = r\cos\theta$ and $y = r\sin\theta$;

$$= a\frac{\int_0^{\pi} (1+\cos\theta)^2 \sin\theta\cos\theta\cos\theta/2\,d\theta}{\int_0^{\pi} (1+\cos\theta)\sin\theta\cos\theta/2\,d\theta}$$

on substituting $r = a(1 + \cos\theta)$;

$$= 2a \frac{\int_0^\pi \cos^6 (\theta/2) \sin (\theta/2) (2 \cos^2 \theta/2 - 1)\, d\theta}{\int_0^\pi \cos^4 (\theta/2) \sin (\theta/2)\, d\theta}$$

as $\cos \theta = 2 \cos^2 \theta/2 - 1$ and $\sin \theta = 2 \sin (\theta/2) \cos \theta/2$;

$$= 2a \frac{\left[-\frac{4}{9} \cos^8 \frac{\theta}{2} + \frac{2 \cos^7 \theta/2}{7} \right]_0^\pi}{\left[-\frac{2}{5} \cos^5 \theta/2 \right]_0^\pi}$$

$$= 2a \frac{\left(\frac{4}{9} - \frac{2}{7} \right)}{2/5} = \frac{50a}{63}.$$

Ex. 3. Find the centroid of a hemispherical distribution of mass in which the density varies as the nth power of the distance from the centre.

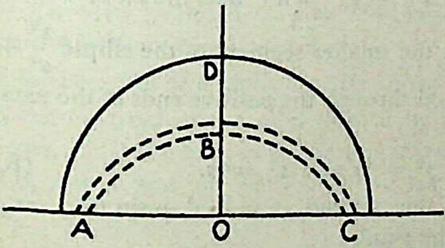

Fig. 66

Dissect the solid hemisphere into a number of thin concentric shells as in the figure. Let the radius of a typical shell ABC be x and its thickness dx. The density ρ of the material of this shell is $k\ x^n$, where k is a constant. Hence, the element of mass $dm = 4\pi x^2\, dx\, k\, x^n$ and its centre of mass is at a distance $\frac{x}{2}$ from O on OD, the normal at O to the plane base, as this shell may be regarded as a hollow hemisphere of radius x (*vide* Ex. 1, page 356).

$\therefore$ The centre of mass of the aggregate of the shells, *i.e.*, the hemisphere lies on OD and its distance $\bar{x}$ from O is given by

$$\bar{x} = \frac{\int_0^a 4\pi x^2\, dx\, k\, x^n \frac{x}{2}}{\int_0^a 4\pi x^2\, dx\, k\, x^n} = \frac{1}{2} \frac{\int_0^a x^{n+3}\, dx}{\int_0^a x^{n+2}\, dx} = \frac{a}{2} \cdot \frac{n+3}{n+4}.$$

Exercises LXXVII.

1. Find the centroids of the following arcs :—

(*i*) the catenary $y = c \cosh \frac{x}{c}$ from the origin to the point (x, y).

(*ii*) the arch of the cycloid

$$x = a(\theta - \sin \theta) \;;\; y = a(1 - \cos \theta).$$

(*iii*) the curve $x^{2/3} + y^{2/3} = a^{2/3}$ between the successive cusps.

(*iv*) the cardioid $r = a(1 + \cos \theta)$.

2. Find the distance of the origin to the centroid of the arc of the curve $4y^2 = x^3$ from $y = -\frac{1}{2}$ to $y = \frac{1}{2}$. (B.Sc. 48 M)

3. Find the centroids of the areas bounded by :

(*i*) $y^2 = 4ax$ and the double ordinate $x = h$.

(*ii*) $x^{2/3} + y^{2/3} = a^{2/3}$ and the axes. (B.A. 47 M)

(*iii*) the smaller segment of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cut off by the chord through the positive ends of the axes of the curve. (B.A. 51 M)

(*iv*) $y^2 = 4x$ and $x^2 = 4y$. (B.Sc. 52 T.U.)

(*v*) the cycloid $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$, its base and the y-axis.

(*vi*) $x^2 = 4y$ and the straight line $y = x$. (B.Sc. Sub. 41)

(*vii*) $y = \sin x$ between $x = 0$ and $x = \pi$.

(*viii*) $y^2 = x^3$ between the origin and $(1, 1)$.

(*ix*) a loop of the curve $y^2(a + x) = x^2(a - x)$. (B.E. 50)

(*x*) $y = (x - 2)(5 - x)$ and the x-axis.

(*xi*) $y = c \cosh \frac{x}{c}$, the x-axis and $x = \pm a$.

(*xii*) a loop of the curve $y^2 = 4x^2 - x^3$.

(*xiii*) the elliptic area enclosed between the minor axis and a double ordinate parallel to it. (B.A. 40 M)

(*xiv*) the area in the first quadrant bounded by the curves $y = 4x^2$ and $y = x^4$. (B.A. Sub. 52)

(*xv*) the area bounded by the axes and the parabola $x^{1/2} + y^{1/2} = a^{1/2}$.

4. Find the centroids of the areas bounded by

(*i*) the cardioid $r = a(1 + \cos\theta)$.

(*ii*) a loop of the curve $r = a\cos 2\theta$.

(*iii*) a loop of the curve $r = a\cos 3\theta$.

5. Find the centroid of the volume of a spherical cap of height h.

6. Find the centroids of the solids formed by the revolution of the following curves :—

(*i*) $y^2 = 4ax$ cut off by the ordinate $x = h$ about the x-axis.

(*ii*) the quadrant of an ellipse about its major axis.

(*iii*) the circle about a tangent line.

(*iv*) $x^2 - y^2 = a^2$ and $x = 2a$ about the x-axis.

(*v*) one arch of the curve $y = \sin x$ about the x-axis.

(*vi*) the area bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4by$ about the axis of x. (B.E. 50)

(*vii*) the area bounded by $y = x^2 - 4x + 6$ about the x-axis between the sections $x = 1$ and $x = 4$. (B.E. 52)

7. Find the centroid of a right circular cone.

8. Find the centroid of the solid formed by the revolution about the y-axis, of (*i*) the area in the first quadrant bounded by the lines $y = 0$, $x = a$ and the parabola $y^2 = 4ax$, (*ii*) the area bounded by $x^2 - y^2 = 1$, $y = 0$, $y = 1$.

9. Find the centroids of the surfaces formed by the revolution of

(*i*) the parabola $y^2 = 2ax$ cut off by $x = c$ about the x-axis.

(*ii*) the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ about the axis of y.

(*iii*) one loop of $r^2 = a^2\cos 2\theta$ about the initial line.

10. Find the centre of mass of a hemispherical distribution of mass in which density varies (*i*) inversely as the distance from the centre ; (*ii*) directly as the distance from the axis of symmetry (B.Sc. Sub. 44) ; (*iii*) directly as the square of the distance from the plane base (B.Sc. Sub. 44) ; (*iv*) directly as the distance from the base. (B.A. 42 M)

11. Find the centre of gravity of a solid right circular cone, the density at any point being proportional to its distance from the base.

12. The density of a circular lamina at any point varies as the distance of the point from a fixed tangent to the circle. Show that the centre of mass of the lamina is at a distance $\frac{5a}{4}$ from the fixed tangent, a being the radius of the circle.

§ 63·6. Pappus' theorems.

1. *If an arc of a plane curve rotates about an axis in its own plane, the area of the surface is equal to the perimeter of the arc multiplied by the length of the path described by the centroid of the arc.*

2. *If a plane area rotates about an axis in its own plane, the volume of the solid generated is equal to the product of the area and the length of the path described by the centroid of the area.*

Let the axis of rotation be taken as the axis of x and let it not cut the curve. Let us suppose that the curve makes a complete revolution.

(1) Let PQ be an element of arc ds and let it revolve about the x-axis. Let P be the point (x, y). P describes a circle centre L and radius y. The element of surface generated by arc $PQ = 2\pi y ds$. Hence the whole surface generated

$$= \int 2\pi y ds = 2\pi \int y ds = 2\pi \bar{y} \times L$$

as $\int y ds = \bar{y} \int ds = \bar{y} L$ from § 63·2,

where L is the perimeter of the given arc and $\bar{y}$ is the ordinate of the centroid of the arc. The length of the path described by the centroid of the arc is $2\pi\bar{y}$. Hence the theorem.

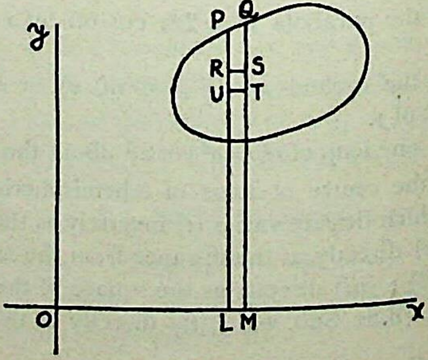

Fig. 67

(2) Let δA be an element of area $RSTU$, where the ordinate of R is y. If this area revolves obout Ox, the element of the volume of the solid generated $= 2\pi y\, dA$.

The total volume of the solid generated $= 2\pi \int y \, dA = 2\pi A \bar{y}$, where A is the total area and $\bar{y}$ is the ordinate of the centroid of the area as $\int y \, dA = \bar{y} \int dA = A\bar{y}$ from § 63·3, (1). The length of the path described by the centroid of the area is $2\pi \bar{y}$. Hence the result follows.

Note.—The above results are evidently true if the arc or the area does not make a complete revolution but only turns through an angle whose circular measure is a. We have only to replace in the above argument the factor 2π by a.

Example. Find the positions of the centroids of the arc and area of a semicircle.

(1) Let the arc of a semicircle revolve about its bounding diameter. The surface of the sphere generated $= 4\pi a^2$.

The centroid of the arc lies on the middle radius by symmetry. Let its distance from the centre be $\bar{y}$. The arc of the semicircle $= \pi a$ and the length of the path described by the centroid of the arc $= 2\pi\bar{y}$.

By Pappus' theorem (1), $4\pi a^2 = \pi a \times 2\pi\bar{y}$.

Hence $\bar{y} = 2a/\pi$.

(2) When the area revolves about the bounding diameter the volume of the sphere generated $= 4\pi a^3/3$. The area of the semicircle $= \pi a^2/2$. The centroid of the area lies on the middle radius by symmetry. Let its distance from the centre be $\bar{y}$. The length described by the centroid $= 2\pi\bar{y}$. By Pappus' theorem (2),

$$\frac{4\pi a^3}{3} = \frac{\pi a^2}{2} \cdot 2\pi\bar{y}. \quad \text{Hence } \bar{y} = \frac{4a}{3\pi}.$$

Exercises LXXVIII.

1. A circle of radius r rotates about an axis in its own plane at distance c ($> r$) from its centre ; find the volume and surface area generated of the anchor ring.

2. Apply Pappus' theorems to find the surface and the volume of a frustum of a cone in terms of its height and the radii of its end circular sections.

3. A triangle, of area Δ, revolves about an axis in its plane, the perpendicular distances of its vertices from the line being a, β, γ.

Show that the volume generated is $\frac{2\pi\Delta}{3}(\alpha + \beta + \gamma)$.

4. A circle of radius a rotates about a tangent. Find the volume formed. (B.E. 49)

5. An ellipse revolves about the tangent at the end of the major axis. Find the volume of the solid generated.

6. The lemniscate $r^2 = a^2 \cos 2\theta$ revolves about the tangent at the pole. Show that the volume generated is $\frac{\pi^2 a^3}{4}$.

7. Find the area of the surface generated by the revolution of the cycloid $x = a(\theta - \sin\theta)$; $y = a(1 - \cos\theta)$ about the line $y = 0$.

8. A square with a semicircle on one of its sides rotates about the opposite side. Find the volume generated.

9. Use Pappus' theorem to find the volume of the solid generated by revolving $r = a(1 + \cos\theta)$ about its axis.

§ 64·1. Moment of Inertia.

Def. If m be the mass of a particle and r be its distance from a given line, the product mr^2 is called its *moment of inertia* about the given line. For a lamina or a rigid body, the sum of similar expressions for all particles of the body, *i.e.*, $\Sigma\, mr^2$ is called the moment of inertia of the body about the given line.

If a system consists of only a finite number of isolated particles, $\Sigma\, mr^2$ is calculated by addition. But if the system consists of a continuous distribution of matter as in the case of a lamina or a rigid body, the number of particles is infinitely great and sum $\Sigma\, mr^2$ is effected by integration. If dm be an element of mass of a body at distance r from a given straight line, then the moment of inertia of the body about the line is $\int r^2 dm$ taken throughout the body. If this integral be written as Mk^2, where M is the mass of the body, k is called the radius of gyration of the body about the given axis.

We give below two theorems which often facilitate calculation of moments of inertia.

§ 64·2. Theorem of parallel axes.

If I denotes the $M.I.$ of a body about an axis through its centre of inertia and I' its $M.I.$ about any parallel axis, then

$I' = I + Md^2$, where M is the mass of the body and d the distance between the parallel axes.

Consider the case of a plane lamina. Let its moment of inertia about the axis Oy' be I_y' and about the parallel axis Gy through G, the centre of inertia be I_y.

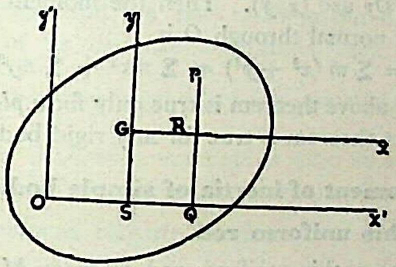

Fig. 68

Take any origin O on Oy' and draw Ox', perpendicular to Oy' in the plane of the lamina. Let Gx be the parallel through G to Ox'.

Consider a typical element of the lamina of mass m at P whose co-ordinates referred to the axes Ox', Oy' are (x', y'). Let G be $(\bar{x}, \bar{y})$ referred to Ox', Oy' and P be (x, y) with reference to Gx, Gy. Then $x' = OQ = OS + SQ = \bar{x} + x$. Similarly $y' = \bar{y} + y$.

By definition, $I_y' = \Sigma\, mx'^2 = \Sigma\, m(\bar{x} + x)^2$

$$= \Sigma\, mx^2 + 2\,\Sigma\, mx \,.\, \bar{x} + \Sigma\, m\bar{x}^2$$

$$= I_y + 2\,\bar{x}\,\Sigma\, mx + \bar{x}^2\,\Sigma\, m$$

as, in the last two terms, $\bar{x}$ being constant for each term can be removed out of the summation sign.

Now, $\Sigma\, mx / \Sigma\, m$ is the abscissa of the centre of gravity of the lamina referred to G as origin and hence vanishes.

$\therefore\ I_y' = I_y + M\bar{x}^2$ where M is the mass of the lamina.

Similarly the proposition is true for parallel axes Ox' and Gx; *i.e.*, $I_x' = I_x + M\bar{y}^2$.

The *M.I.* of the lamina about an axis through O perpendicular to its plane

$$= \Sigma\, mOP^2 = \Sigma\, m(x'^2 + y'^2) = \Sigma\, m[(\bar{x} + x)^2 + (\bar{y} + y)^2]$$

$$= \Sigma\, m(\bar{x}^2 + \bar{y}^2) + 2\bar{x}\,\Sigma\, mx + 2\bar{y}\,\Sigma\, my + \Sigma\, m(x^2 + y^2)$$

$= (\bar{x}^2 + \bar{y}^2)\,M + \Sigma\, m(x^2 + y^2)$ as $\Sigma\, mx$ and $\Sigma\, my$ vanish

$= M.OG^2 + M.I.$ about a parallel axis through G. Thus the theorem is proved for lines normal to the lamina.

The argument can be extended on similar lines to the case of any rigid body in three dimensions.

§ 64·3. Theorem of perpendicular axes.

If I_x and I_y be the moments of inertia of a plane lamina about two rectangular axes Ox, Oy in its plane, its moment of inertia about an axis through O perpendicular to its plane is I_x+I_y.

Consider a typical element of mass m at P whose co-ordinates referred to Ox, Oy are (x, y). Then the moment of inertia of the lamina about a normal through O is

$$\Sigma\, m\, OP^2 = \Sigma\, m\,(x^2+y^2) = \Sigma\, mx^2 + \Sigma\, my^2 = I_y + I_x.$$

Note.—The above theorem is true only for *a plane lamina* while the parallel axes theorem is true for any rigid body.

Moment of inertia of simple bodies.

§ 64·4. Thin uniform rod.

Let AB be a thin uniform rod of mass M and length $2a$. Let O be its midpoint and Oy the perpendicular to the rod through O. Let us calculate the $M.I.$ about Oy.

Take P and Q two neighbouring points at distances x and $x+\delta x$ from O so that the length of the element PQ of the rod is δx.

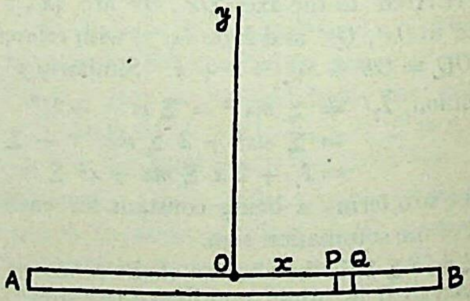

Fig. 69

Its mass is $\rho\, dx$ (neglecting the thickness of the rod) where ρ is the line density (mass per unit length) of the rod.

Hence its $M.I.$ about $Oy = \rho\, dx \,.\, x^2$.

The $M.I.$ of rod about $Oy = \Sigma\, \rho\, dx \,.\, x^2 = \rho \int_{-a}^{+a} x^2\, dx$

$$= Ma^2/3 \text{ as } \rho = M/2a.$$

Cor. 1. $M.I.$ of the rod about an axis through the end $A \perp AB = \rho \int_0^{2a} x^2\, dx = \frac{8a^3}{3}\rho = \frac{4Ma^2}{3}$. (This result also follows by the parallel axes theorem on Moments of Inertia.)

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Cor. 2. *M.I.* of the rod about an axis through O inclined at θ to $AB = \dfrac{M}{2a}\displaystyle\int_{-a}^{+a} x^2 \sin^2\theta \, d\theta = \dfrac{Ma^2 \sin^2\theta}{3}$.

Cor. 3. If the rod be not uniform but the density at P be a function of its distance x from O, then the *M.I.* about Oy is $\int_{-a}^{+a} \rho \, dx \, . \, x^2$, where ρ is given by the law of density.

Cor. 4. The *M.I.* of a rectangle of sides $2a$ and $2b$ is $Ma^2/3$ about an axis through its centre of gravity parallel to the $2b$ edge.

§ 64·5. Uniform elliptic lamina.

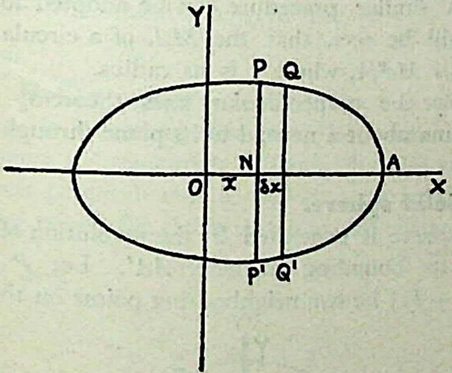


Fig. 70

Let P (x, y) and Q $(x + \delta x, y + \delta y)$ be two neighbouring points on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Divide the lamina into strips such as $PP'Q'Q$ perpendicular to the major axis, the width of the strip being δx. The mass δm of the strip is $2y \, \delta x \, \rho$, where ρ is the surface density (mass per unit area) of the lamina. This strip is practically in the form of a thin uniform rod of length $2y$ and its *M.I.* about the axis OA is $2 \, \rho y \, dx \, y^2/3$ by § 64·4.

Summing up for all such strips, the *M.I.* of the lamina about its major axis is :

$$\int_{-a}^{+a} 2 \, \rho y^3 \frac{dx}{3} = \frac{4\rho}{3}\int_0^a y^3 \, dx \text{ by symmetry}$$

$$= \frac{4\rho b^3}{3} \int_0^a \left(1 - \frac{x^2}{a^2}\right)^{3/2} dx$$ on substituting for y from the equation of the ellipse.

Putting $x = a \sin\theta$, the *M.I.* is $\dfrac{4\rho a}{3} b^3 \displaystyle\int_0^{\pi/2} \cos^4\theta \, d\theta$

$$= \frac{4\rho ab^3}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi ab^3 \rho}{4} = \frac{Mb^2}{4}$$ as M (the mass of the lamina) is $\pi ab\rho$.

Similarly, the *M.I.* of the elliptic lamina about its minor axis can be got by dividing it into strips parallel to the major axis and is $Ma^2/4$.

Cor. 1. A similar procedure can be adopted for a circular lamina. It will be seen that the *M.I.* of a circular disc about any diameter is $Ma^2/4$, where a is its radius.

Cor. 2. By the perpendicular axes theorem, the *M.I.* of a circular lamina about a normal to its plane through its centre is $Ma^2/2$.

§ 64·6. Solid sphere.

A solid sphere is generated by the revolution of a semicircle APA' about its bounding diameter AA'. Let P (x, y) and Q $(x + \delta x, y + \delta y)$ be two neighbouring points on the generating

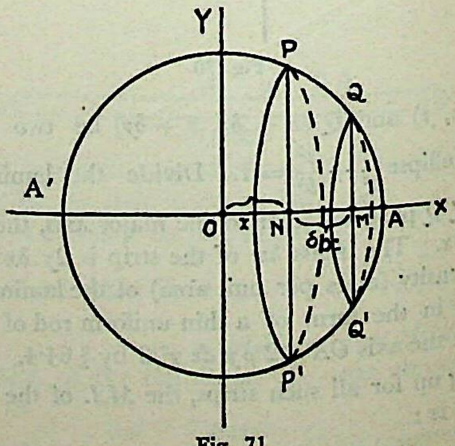


Fig. 71

circle $x^2 + y^2 = a^2$. Divide the sphere into such slices as $PP'Q'Q$ bounded by circular sections through P and Q at distances x and $x + \delta x$ from O. The mass δm of this slice is $\pi y^2 \, \delta x \, \rho$, where ρ is

the volume density (mass per unit volume), of the sphere. Since the typical mass is ultimately in the form of a circular disc of radius y, its *M.I.* about $OX = \pi y^2\,\delta x\,\rho\,y^2/2$ by § 64·5, Cor. 2. Hence the *M.I.* of the sphere about the diameter OX is

$$\frac{\pi\rho}{2}\int_{-a}^{+a} y^4\,dx = \pi\rho\int_0^a (a^2 - x^2)^2\,dx \text{ by § 52 and as } y^2 = a^2 - x^2\,;$$

$$= \pi\rho\int_0^a (a^4 - 2a^2x^2 + x^4)\,dx = \pi\rho\left[a^4x - \frac{2a^2x^3}{3} + \frac{x^5}{5}\right]_0^a$$

$$= \frac{8\pi\rho a^5}{15} + \frac{2}{5}Ma^2 \text{ as } \frac{4\pi\rho a^3}{3} = M \text{ (the mass of the sphere).}$$

Cor. The *M.I.* about a tangent line $= \dfrac{7\,Ma^2}{5}$ by the parallel axes theorem.

§ 64·7. Hollow sphere. By revolving a semi-circular arc ABA' about its bounding diameter $A'A$ taken as the x-axis, a hollow sphere is generated. Divide the hollow sphere into circular bands perpendicular to OX such as $PQQ'P'$. Let the point $P\,(x, y)$ be on the circle whose arcual distance from A is s and let $A\hat{O}P = \theta$.

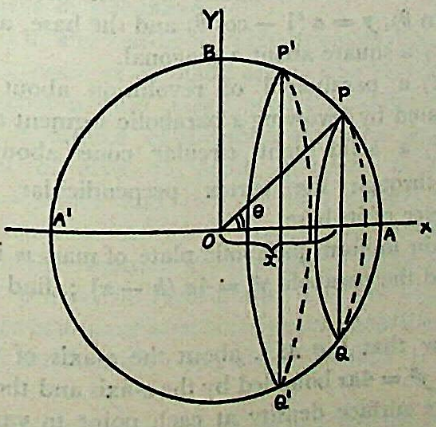


Fig. 72

Let P' be a neighbouring point on the circle so that arc PP' is δs and $P\hat{O}P'$ is $\delta\theta$. The element of the surface $PQQ'P'$

is $2\pi\, y\, \delta s$ (*vide* § 62) and its mass is $2\pi\, y\, \delta s\, \rho$, where ρ is the surface density of the sphere. This mass is in the form of a circular ring of radius y and its *M.I.* about *OX* is $2\pi\, y ds\, \rho y^2$. (In a uniform circular ring, each element is at the same distance from its centre and hence the *M.I.* of every element of the arc about the axis through the centre perpendicular to the plane is its mass multiplied by the square of the radius. Summing up for all the elements, the *M.I.* of the ring is Ma^2, where M and a are its mass and radius.) Hence the *M.I.* of the hollow sphere about the diameter *OX*

$$= 2\pi\, \rho \int y^3\, ds = 2\rho\, a^4 \int_0^{\pi} \sin^3\theta\, d\theta$$

as $y = a \sin\theta$ and $s = a\theta$

$$= 4\pi\rho a^4 \int_0^{\pi/2} \sin^3\theta\, d\theta \text{ by Ex. 5, page 283}$$

$$= 4\pi\rho a^4\, \frac{2}{3} = \frac{2\, Ma^2}{3} \text{ as } 4\pi a^2\, \rho = M.$$

Exercises LXXIX.

1. Find the moments of inertia of :

(*a*) a uniform right circular cylinder about (*i*) its axis, (*ii*) a diameter of one of its ends.

(*b*) a lamina bounded by one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ and the base, about the base.

(*c*) a square about a diagonal.

(*d*) a paraboloid of revolution about its axis (*i.e.*, a solid generated by revolving a parabolic segment about its axis).

(*e*) a solid right circular cone about (*i*) its axis, (*ii*) a line through the vertex perpendicular to the axis, (*iii*) a diameter of its base. (B.Sc. 56 M)

2. A thin uniform parabolic plate of mass m is bounded by the y-axis and the parabola $y^2 = 4a(h - x)$; find its *M.I.* about the y-axis. (An.U. 40)

3. Show that the *M.I.* about the x-axis of the portion of the parabola $y^2 = 4ax$ bounded by the x-axis and the latus rectum, supposing the surface density at each point to vary as the cube of the abscissa is $12\, Ma^2/11$, where M is the mass of the lamina. (B.A. 36 M)

4. Find the *M.I.* of a lune of a sphere of angle α about the bounding diameter. (B.A. 46 M)

5. A uniform ring-shaped lamina bounded by concentric circles of radii 5 cm. and 7 cm. is cut into two equal parts along a diameter. Find the *M.I.* of either part about the line of section, if the mass of the entire piece is 32 gm. (B.A. Sub. 47)

6. For a solid hemisphere whose density at any point is proportional to the distance from the base, find the radius of gyration about the axis of symmetry. (B.A. 42 M)

7. Find the moment of inertia of a lamina of uniform thickness of the form $r = a\,(1 + \cos\theta)$ about an axis perpendicular to its plane through the pole.

8. Find the moment of inertia of a circular plate in which the density varies as the distance from the centre about the axis through the centre perpendicular to the plate.

9. Find the *M.I.* of a paraboloid of revolution bounded by $x = l$ about a tangent line at the vertex. (B.E. 46)

10. Find the *M.I.* of a triangular lamina about a line through a vertex parallel to the opposite side. (B.A. 53 M)

11. Find the *M.I.* about its axis of the solid of revolution by rotating the curve $y = e^x \cos x$ $(0 < x < 1)$ about the x-axis. (B.A. 44 M)

12. Find the *M.I.* of a hollow circular cylinder about its axis, the external and internal radii being R and r respectively. (B.A. 52)

§ 65. Centre of pressure.

It is proved in any text-book of Hydrostatics that the pressure intensity at any point of a plane area exposed to the action of a heavy homogeneous fluid is ρgz, where z is the depth of the point below the surface of the fluid and ρ its density. If dA be an element of area at depth z below the surface, the thrust on $\delta A = \rho gz dA$ and the resultant thrust on the area $= \rho g \int z dA$ taken over the whole area. The point of a plane surface at which the resultant pressure acts is called the centre of pressure. It is proved in Hydrostatics that the depth $\bar{z}$ of the centre of pressure of a plane area acted on by a heavy homogeneous fluid is given by the formula $\bar{z} = \int z^2\, dA \,/ \int z\, dA$.

Examples.

Ex. 1. Find the centre of pressure of a vertical circular area of radius a wholly immersed with its centre at depth h.

Let O be the centre of the circle at depth h below the surface. Choose O as the origin and the downward vertical through O as

the x-axis. Divide the area into strips like $PQQ'P'$ parallel to the surface of the fluid. Let P be the point (x, y) on the circle. Let the width of the typical strip $PQQ'P'$ be dx. Then $\delta A = 2y dx$. The pressure intensity at every point of the strip is $\rho g\,(h+x)$ and hence pressure over the strip $= \rho g\,(h+x)\,2y dx$.

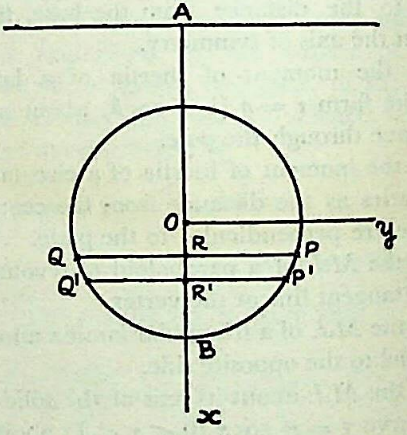


Fig. 73

By symmetry, the centre of pressure lies on OB.

$$\text{Its depth } \bar{x} \text{ (below } O) = \frac{\int_{-a}^{+a} \rho g\,(h+x)\,2y\,x\,dx}{\int_{-a}^{+a} \rho g\,(h+x)\,2y\,dx}$$

$$= \frac{\int_{-a}^{+a} (h+x)\,x\sqrt{a^2-x^2}\,dx}{\int_{-a}^{+a} (h+x)\sqrt{a^2-x^2}\,dx} \quad \text{as } y = \sqrt{a^2-x^2} \text{ from the equation of the circle.}$$

Putting $x = a\cos\theta$; $dx = -a\sin\theta\,d\theta$,

$$\bar{x} = a \cdot \frac{\int_0^{\pi} (h + a\cos\theta)\cos\theta\sin^2\theta\,d\theta}{\int_0^{\pi} (h + a\cos\theta)\sin^2\theta\,d\theta}$$

$$= \frac{ah\left[\frac{\sin^3\theta}{3}\right]_0^{\pi} + a^2\int_0^{\pi}\cos^2\theta\sin^2\theta\,d\theta}{h\int_0^{\pi}\sin^2\theta\,d\theta + a\left[\frac{\sin^3\theta}{3}\right]_0^{\pi}}$$

$$=\frac{a^2 \cdot 2 \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}}{h \cdot 2 \cdot \frac{1}{2} \cdot \frac{\pi}{2}}=\frac{a^2}{4h}.$$

Hence the depth of C.P. below the surface $= h + \frac{a^2}{4h}$.

Ex. 2. Find the centre of pressure of a rectangular lamina immersed in a liquid vertically with one side in the surface.

Let $ABCD$ be the rectangular lamina with AB in the surface and let the sides AD and AB be a and b respectively.

Dissect the lamina into strips like $PQQ'P'$ parallel to the surface. Let AP be x and let the width PP' of the strip be dx.

The area of this strip $dA = b\,dx$.

The pressure intensity over this strip $= \rho g x$.

Hence the pressure over this strip $= b\,dx\,g\,\rho x$ and acts at the midpoint of PQ. The centre of pressure of every such strip and

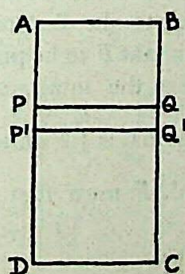


Fig. 74

therefore of the lamina lies on the line joining the midpoints of AB and CD.

Its depth below $AB = \dfrac{\int_0^a g\,\rho\,b\,x^2\,dx}{\int_0^a g\,\rho\,b\,x\,dx} = \dfrac{\left[\frac{x^3}{3}\right]_0^a}{\left[\frac{x^2}{2}\right]_0^a} = \frac{2}{3}a.$

Exercises LXXX.

Find the centre of pressure in the following cases :—

1. A rectangular lamina, sides a and b, immersed vertically in a liquid, with the sides a parallel to the surface, the upper one being at depth h.

2. A triangle immersed with vertex in the surface and opposite side horizontal.

3. A triangle immersed in a liquid with a side in the surface.

4. A parallelogram immersed in a liquid with one side in the surface.

5. A semicircle with its bounding diameter in the surface.

6. An ellipse with its major axis vertical and one vertex in the surface.

§ 66. Work done by a force.

When the force F is constant, the work done by it is $F \times s$ where s is the distance moved through by the point of application of the force in the direction of the force.

When the force varies with time t, we proceed as follows to calculate the work done. Let P, the point of application of the force move along a curve BC. At time t, let the arcual distance of P measured from a fixed point A on the curve be s. When the point P moves through a further small distance δs in time δt, the elementary work done is, to the first order of small quantities, Fds (for during δt, we can take F to be practically constant though it actually varies). Hence the total work done as the point of application moves from B to C is $\int_{s_1}^{s_2} F\,ds$, where $\widehat{AB}=s_1$ and $\widehat{AC}=s_2$.

To calculate the integral, F must first be expressed as a function of s.

Examples.

Ex. 1. Find the work done by an elastic string, of natural length a, in stretching it from a length b to c $(c > b > a)$.

Let T be the tension of the elastic string when its length is x and let λ be the modulus of elasticity.

By Hooke's Law, $T = \dfrac{\lambda (x-a)}{a}$.

Work done in stretching the string from length b to c

$$= \int_b^c T\,dx = \int_b^c \frac{\lambda(x-a)}{a}\,dx = \frac{\lambda}{2a}\Big\{(x-a)^2\Big\}_b^c$$

$$= \frac{\lambda}{2a}(c-b)(b+c-2a)$$

= Extension × Arithmetic mean of the initial and final tensions.

Ex. 2. A volume v of gas at a pressure p is contained in a cylindrical vessel ; if it be allowed to expand so that the length alters from x_0 to x_1, the temperature remaining constant, show that the work done is $pv \log (x_1/x_0)$.

If it expands adiabatically, *i.e.*, so that no heat passes into or out of the gas and the relation between the pressure p and the volume v is therefore $pv^\gamma =$ constant, show that corresponding work is $\dfrac{pv}{\gamma-1}\left[1-\left(\dfrac{x_0}{x_1}\right)^{\gamma-1}\right]$.

If P is the pressure when the length occupied by the gas is x, $PxA = px_0A$, where A is the section of the cylinder. When the length changes from x to $x+\delta x$, the work done $= \int PdV$, (where V is the volume of gas $= Ax$).

$$= \int_{x_0}^{x_1} \frac{px_0}{x} A dx = pAx_0 \log (x_1/x_0) = pv \log (x_1/x_0) \text{ as } v = Ax_0.$$

In the second case, $PV^\gamma = pv^\gamma$, *i.e.*, $px^\gamma = px_0{}^\gamma$.

$$\text{The work done} = \int PdV = \int_{x_0}^{x_1} p\,\frac{x_0{}^\gamma}{x^\gamma} A\, dx$$

$$= \frac{pAx_0{}^\gamma}{\gamma-1}\left[\frac{1}{x_0{}^{\gamma-1}} - \frac{1}{x_1{}^{\gamma-1}}\right] = \frac{pv}{\gamma-1}\left[1-\left(\frac{x_0}{x_1}\right)^{\gamma-1}\right]$$

as $Ax_0 = v$.

Exercises LXXXI.

1. An elastic string, whose natural length is 6 inches, has a length 12 inches when it supports a mass of 5 lb. Find the work done in stretching it from its natural length to 20 inches.

2. A spiral spring requires a force of 1 lb. weight to stretch it one inch. How much work is done in stretching it 3 inches more ?

3. An elastic cord, natural length 10 inches, can be kept stretched to a length of 15 inches by a force of 5 lb. weight ; find the amount of work done in stretching it from a length of 12 inches to 15 inches.

4. A quantity of air with initial volume of 200 c. ft. and pressure of 15 lb. per square inch is compressed to 80 lb. per square inch. Find the work done if the isothermal law ($pv = c$) holds good.

5. Gas at pressure of 15 lb. per square inch is compressed from 200 c. ft. to 50 c. ft. Find the work done if the adiabatic law, $pv^{\gamma} =$ constant, holds good.

§ 67. **Compound interest law.**

Suppose a sum P of money to be invested at compound interest at the rate of $r\%$ per annum, interest payable n times a year at equal intervals of time. After the first payment of interest, the amount $= P + \frac{P}{100}\frac{r}{n} = P\left(1 + \frac{r}{100n}\right)$. This amount is the principal at the beginning of the second period. The amount at the end of the second period is therefore $P\left(1 + \frac{r}{100n}\right)^2$ and so on. The amount at the end of t years, *i.e.*, nt such periods

$$= P\left(1 + \frac{r}{100n}\right)^{nt}.$$

If $n \to \infty$, *i.e.*, the interest is added to the principal every instant continuously, the amount A at the end of t years is given by

$$A = \underset{n\to\infty}{\text{Lt}}\, P\left(1 + \frac{r}{100n}\right)^{nt} = P \underset{n\to\infty}{\text{Lt}} \left\{\left(1 + \frac{r}{100n}\right)^{\frac{100n}{r}}\right\}^{\frac{rt}{100}}.$$

$$= P\, e^{rt/100} \text{ as } \underset{n\to\infty}{\text{Lt}} \left(1 + \frac{r}{100n}\right)^{\frac{100n}{r}} = e.$$

Hence $\dfrac{dA}{dt} = \dfrac{Pr}{100}\, e^{rt/100} = \dfrac{r}{100} A.$

Thus $\dfrac{dy}{dt} = ky$, where k is a constant, is spoken of as the compound interest law.

Cor. 1. If t increases in arithmetic progression, y increases in geometrical progression.

2. The present value of a sum $Pe^{rt/100}$ due t years hence is P. The present value of one rupee due t years hence, is $e^{-rt/100}$, *i.e.*, $\left(\dfrac{1}{e^{rt/100}}\right)$ of a rupee.

Two examples can be given where the compound interest law functions in nature.

(*i*) Newton's law of cooling : The rate at which the temperature $T°$ of a hot body in a room of constant temperature is

falling is proportional to the excess of the body's temperature above that of the room, *i.e.*, $\frac{dT}{dt} = -k(T - c)$. If θ represents the excess of temperature, $\frac{d\theta}{dt} \doteq -k\theta$.

(*ii*) The variation in pressure due to change of altitude is given by $\frac{dp}{dh} = -kp$, where p is the pressure at height h.

Exercises LXXXII.

1. Find the compound interest on a sum of Rs. 500 for 10 years at 6 % per annum, interest being added to the principal (*i*) once a month, (*ii*) every instant.

2. Show that a sum of money accumulating at the rate of r % p.a. interest being compounded annually will double itself in approximately $\left(\frac{100}{r} + \frac{1}{2}\right) \log 2$ years. (B.A. Sub. 37)

3. Assuming the density P of the atmosphere at any given height varies as the pressure p, show that the pressure at height z from the surface of the earth is given by $p = p_0 e^{-gP_0 z/p_0}$, where p_0 is the pressure and P_0 the density of the atmosphere at the earth's surface. (B.A. Sub. 33)

4. A spherical metallic ball of radius 2 cm. expands under heat. If the coefficient of linear expansion of the metal is ·00002, find the increase in volume when the temperature is raised from 0° C. to 20° C. (B.A. Sub. 35)

5. A tree in a period of 20 years of its growth is found to grow in any one year to 1·02 times its height in the previous year. If it was 3 ft. high at the end of the third year of the period, how high will it be at the end of the fifteenth year? (B.A. Sub. 36)

CHAPTER XV

DIFFERENTIAL EQUATIONS

§ 68·1. Definitions.

A *differential equation* is an equation in which differential coefficients occur.

Differential equations are of two types :—(*i*) *ordinary* and (*ii*) *partial*.

An *ordinary* differential equation is one in which a single independent variable enters, either explicitly or implicitly. For example,

$$\frac{dy}{dx}=2\sin x,\quad \frac{d^2x}{dt^2}+m^2x=0$$

$$\text{and } x^2\frac{d^2y}{dx^2}+2xy\frac{dy}{dx}+y=\sin x$$

are ordinary differential equations.

A *partial* differential equation is one in which at least two (two or more) independent variables occur and the partial differential coefficients occurring in them have reference to any one of these variables.

For example, $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=z$

$$\text{and } \frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2}=\left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2$$

are partial differential equations.

The *order* of an ordinary differential equation is the order of the highest derivative occurring in it.

If a differential equation can be expressed as a polynomial equation in the derivatives, the degree of the highest derivative, when the differential coefficients are cleared of radicals and fractions, is called the *degree* of the equation.

The term 'degree' is not applied to differential equations which cannot be expressed as polynomial equations in the derivatives.

Thus, in $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}=a\dfrac{d^2y}{dx^2}$, the order and degree are

both two while in $2\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=x$, the order is two and the degree is one.

§ 68·2. **Solutions of differential equations.**

A *solution* or *integral* of a differential equation is a relation that exists between the variables, (without their differential coefficients) by means of which and the derivatives obtained therefrom, the equation is satisfied. This solution is also called the *primitive* of the differential equation.

Consider, for example, the differential equation $\frac{d^2y}{dx^2}+n^2y=0$. A solution is $y=A\cos nx$, where A is an arbitrary constant, for $\frac{d^2y}{dx^2}=-An^2\cos nx$ and hence

$$\frac{d^2y}{dx^2}+n^2y=-An^2\cos nx+An^2\cos nx=0.$$

Similarly, $y=B\sin nx$ where B is an arbitrary constant, is also a solution. More generally, $y=A\cos nx+B\sin nx$ is a solution.

The solution, in which the number of arbitrary constants occurring is the same as the order of the equation is called the *general solution* or *complete integral*.

Thus $y=A\cos nx+B\sin nx$ is the general solution of $\frac{d^2y}{dx^2}+n^2y=0$. By giving particular values to the constants occurring in the general solution, we get *particular* solutions or integrals. For example, $y=2\cos nx-\sin nx$ is a particular integral of the above equation.

§ 68·3. **Formation of differential equations.**

We shall consider the derivation of differential equations given their solutions.

Suppose $y=A\sin x+B\cos x$, (1)

where A and B are arbitrary constants. Our object is to form the differential equation whose solution is (1).

Differentiating twice,

$$\frac{dy}{dx}=A\cos x-B\sin x\,;$$

$$\therefore\ \frac{d^2y}{dx^2}=-A\sin x-B\cos x=-y.$$

On eliminating A and B from (1),

$$\frac{d^2y}{dx^2}+y=0 \text{ is the differential equation sought.}$$

Thus, to eliminate the number of constants from a given solution, we differentiate it as often as is necessary and then eliminate the constants. The differential equation is thus obtained.

Consider $\phi(x, y, a)=0$ (1)

where a is an arbitrary constant.

Differentiating (1) with respect to x,

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0. \tag{2}$$

Eliminating a between (1) and (2), we get a differential equation of the form $f\left(x, y, \frac{dy}{dx}\right)=0.$ (3)

Equation (3) includes all the values of y arising from (1) corresponding to the various values of a and therefore represents the differential equation of which (1) is the general solution.

Examples.

Ex. 1. Form the differential equation by eliminating α and β from $(x-\alpha)^2+(y-\beta)^2=r^2.$ (1)

Differentiating with respect to x twice,

$$(x-\alpha)+(y-\beta)\frac{dy}{dx}=0. \tag{2}$$

$$1+(y-\beta)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0. \tag{3}$$

Eliminating α between (1) and (2),

$$(y-\beta)^2\left[\left(\frac{dy}{dx}\right)^2+1\right]=r^2. \tag{4}$$

Eliminating β between (3) and (4),

$$r^2\left(\frac{d^2y}{dx^2}\right)^2=\left[1+\left(\frac{dy}{dx}\right)^2\right]^3.$$

Note.—We have, in fact, formed the differential equations of circles of given radius.

Ex. 2. Form the differential equation that represents all parabolas each of which has a latus rectum $4a$ and whose axes are parallel to the x-axis.

If the vertex of any parabola of the system be (α, β), the equation of the parabola is

$$(y-\beta)^2 = 4a(x-\alpha). \tag{1}$$

We have to eliminate α and β from this equation and the derived equation.

Differentiating twice with repect to x,

$$(y-\beta)\frac{dy}{dx} = 2a. \tag{2}$$

$$(y-\beta)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0. \tag{3}$$

Eliminating β between (2) and (3),

$$2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0.$$

This is the required differential equation.

Exercises LXXXIII.

1. Eliminate c from $y = cx + c - c^3$.

2. Eliminate a and b from (*i*) $y = ae^{2x} + be^{3x}$

(*ii*) $xy = ae^x + be^{-x}$.

3. Form the differential equation which has $y = a\cos(nx+b)$ as general solution, where a and b are arbitrary constants.

4. Form the differential equation having for its general solution

(*i*) $y = ax^2 + bx$ (B.Sc. 51 T.U.)

(*ii*) $y = ax^3 + bx^2$

where a and b are arbitrary constants.

5. Find the differential equation of all circles passing through the origin and having their centres on the x-axis.

6. Show that the differential equation of all parabolas having their axis of symmetry coincident with the axis of x is

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0.$$ (B.Sc. 51 T.U.)

7. Obtain the differential equations satisfied by the curves (*i*) $y = c(x-c)^2$ (B.Sc. 43 T.U.)

(*ii*) $y = ax\cos\left(\frac{n}{x} + b\right)$

where a and b are parameters. (B.Sc. 51)

8. Obtain the differential equations of

(*i*) rectangular hyperbolas which have the axes of coordinates as asymptotes. (B.Sc. 41 T.U.)

(*ii*) circles which touch the y-axis at the origin. (B.Sc. 42 T.U.)

(*iii*) parabolas having their focus at the origin and axis along the x-axis. (B.Sc. 44 T.U.)

§ 69. Equations of the first order and the first degree.

§ 69·1. Type A :—Variables separable.

Suppose an equation is of the form $f(x)\,dx + F(y)\,dy = 0$. We can directly integrate this equation and the solution is $\int f(x)\,dx + \int F(y)\,dy = c$, where c is an arbitrary constant.

Examples.

Ex. 1. Solve $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0$.

We have $\dfrac{dy}{\sqrt{1-y^2}} + \dfrac{dx}{\sqrt{1-x^2}} = 0$.

Integrating, $\sin^{-1} y + \sin^{-1} x = c$.

Ex. 2. Solve $y\,dx - x\,dy + 3x^2y^2\,e^{x^3}\,dx = 0$. (B.E. 48)

This equation can be written as

$$\frac{y\,dx - x\,dy}{y^2} + 3x^2\,e^{x^3}\,dx = 0.$$

$$d\left(\frac{x}{y}\right) + d\left(e^{x^3}\right) = 0.$$

$\therefore$ The solution is $\dfrac{x}{y} + e^{x^3} = c$.

Ex. 3. The stress p in thick cylinders is given by $r\dfrac{dp}{dr} + 2p = 2c$, where c is a constant. Find p in terms of r. (B.Sc. Sub. 33)

$$r\frac{dp}{pr} = 2(c-p)\;;\; \frac{dp}{c-p} = \frac{2\,dr}{r}.$$

Integrating, $-\log(c-p) = 2\log r - \log A$, where A is an arbitrary constant.

$\therefore$ $(c-p)\,r^2 = A$ is the solution.

Exercises LXXXIV.

Solve the equations :

1. $\dfrac{dy}{dx} = \dfrac{y+2}{x-1}$.

2. $e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$.

3. $x\sqrt{1+y^2} + y\sqrt{1+x^2}\dfrac{dy}{dx} = 0$.

4. $\tan y \sec^2 x \, dx + \tan x \sec^2 y \, dy = 0$.

5. $\sqrt{1+x^2}\, dx + \sqrt{1+y^2}\, dy = 0$.

6. $\sqrt{1-x^2} \sin^{-1} x \, dy + y \, dx = 0$. (B.A. 41 M)

7. $(1-x^2)\dfrac{dy}{dx} + xy = 5x$. (B.A. 46 M)

8. Find the curve whose gradient at any point $P(x, y)$ on it is $\dfrac{x-a}{y-b}$ and which passes through the origin. (B.A. 41 M)

9. The subtangent at any point of a curve is equal to the square of the abscissa. If it passes through (2, 1), find its equation.

10. The subnormal at any point of a curve varies as the abscissa and the curve passes through the points (2, 3) and (– 1, 2). Find the equation of the curve.

11. A particle moves in a straight line with a velocity given by $\dfrac{ds}{dt} = s + 1$, where s is the distance from the starting point measured in feet and the unit of time is one second. Show that the time taken by the particle to traverse a distance of 33 yards is $2 \log_e 10$ seconds. (B.Sc. 53 T.U.)

12. An equation in the theory of stability of an aeroplane is $\dfrac{dv}{dt} = g \cos \alpha - kv$, where g, k and α are constants. Solve the equation given $v = 0$ when $t = 0$. (B.Sc. Sub. 33)

13. The distance x descended by a person falling by means of a parachute satisfies the differential equation

$$\left(\frac{dx}{dt}\right)^2 = k^2\left(1 - e^{-2gx/k^2}\right),$$

where k and g are constants ; and $x = 0$ when $t = 0$; show that $x = \dfrac{k^2}{g} \log \cosh \dfrac{gt}{k}$. (B.Sc. Sub. 34)

14. The angular velocity ω of a heavy flywheel rotating in rough bearings is given by $I\dfrac{d\omega}{dt} = -k\omega^2$, where I and k are constants. If the angular velocity at the end of one minute is 90 % of the initial angular velocity, show that the angular velocity at the end of the next minute will fall to $81\frac{9}{11}$ %.
(B.Sc. Sub. 34)

15. A rocket of weight 10 lbs. is fired vertically upwards starting from rest. The effect of the gradual burning of the charge is to produce an acceleration $\dfrac{15g}{10-t}$ ft./sec.^{2} at time t. The charge is consumed in 5 sec. What is the velocity of the rocket at time t?
(B.Sc. Sub. 54)

16. Solve $\dfrac{dy}{dx} - x\tan(y-x) = 1$. (B.Sc. 54 M)

(*Hint* : put $z = y - x$.)

17. $\dfrac{dy}{dx} + \dfrac{y^2+y+1}{x^2+x+1} = 0$. (B.E. 52)

18. $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$.

19. $\dfrac{dy}{dx} = e^{2x-y} + x^3 e^{-y}$. (B.E. 52)

20. $(x^2 - yx^2)\dfrac{dy}{dx} + (y^2 + x^2y^2) = 0$. (B.Sc. 49 M)

21. A curve passes through the point (2, 1) and its subnormal is always given by $3x + 2$. Find the equation of the curve and the subtangent at the point $x = 3$. (B.A. 55 M)

22. $(2x - y + 1)\,dx + (2y - x + 1)\,dy = 0$.
(B.Sc. Comp. Math. 60)

23. $y - x\dfrac{dy}{dx} = b\left(1 + x^2\dfrac{dy}{dx}\right)$. (B.Sc. Anc. 60)

24. $(e^x + 1)\,y\,dy + (y + 1)\,dx = 0$. (B.Sc. 60)

25. $(x - y)^2\dfrac{dy}{dx} = a^2$. (B.Sc. 61)

(*Hint* : put $z = x - y$.)

26. $\dfrac{dy}{dx} = (4x + y + 1)^2$. (B.Sc. Anc. 61)

§ 69·2. Type B :—Homogeneous equations.

Consider $\frac{dy}{dx}=\frac{f_1(x,y)}{f_2(x,y)}$, where f_1 and f_2 are homogeneous functions of the same degree in x and y.

$\therefore$ $f_1(x,y)$ can be written as $x^n\phi\left(\frac{y}{x}\right)$ and $f_2(x,y)$ as $x^n\Psi\left(\frac{y}{x}\right)$, if n is the degree of homogeneity. If we put $y=vx$, $\frac{dy}{dx}=v+x\frac{dv}{dx}$. The given equation becomes $v+x\frac{dv}{dx}=\frac{\phi(v)}{\Psi(v)}$.

The variable can be separated ; the equation is

$$\frac{dx}{x}+\frac{\Psi(v)\,dv}{v\Psi_{;}(v)-\phi(v)}=0.$$

Integrating, $\log x+\int\frac{\Psi(v)\,dv}{v\Psi(v)-\phi(v)}=c.$

The solution is got by substitution $\frac{y}{x}$ for v after the integration is over.

Examples.

Ex. 1. Solve $y^2+x^2\frac{dy}{dx}=xy\frac{dy}{dx}$.

Put $y=vx$; $\frac{dy}{dx}=v+x\frac{dv}{dx}$.

The equation reduces to $\frac{dv\,(1-v)}{v}+\frac{dx}{x}=0.$

Integrating, $\log v-v+\log x=\log c.$

The solution is $y=c\,e^{y/x}$.

(B.Sc. 50 M)

Ex. 2. $xdy-ydx=\sqrt{x^2+y^2}\,dx.$

Put $y=vx$; $dy=v\,dx+x\,dv.$

The equation reduces to $\frac{dv}{\sqrt{1+v^2}}=\frac{dx}{x}.$

Integrating, $\log(v+\sqrt{1+v^2})=\log x+\log c.$

The solution is $y+\sqrt{x^2+y^2}=cx^2$.

(This can be reduced to the form $c^2x^2=1+2cy$.)

Exercises LXXXV.

Solve the following equations :—

1. $\dfrac{dy}{dx}=\dfrac{x-y}{x+y}$.

2. $(y^2-2xy)\,dx=(x^2-2xy)\,dy$. (B.E. 49)

3. $(x^2+y^2)\dfrac{dy}{dx}=xy$. (B.Sc. 45 M)

4. $(x+y)^2\,dx=2x^2\,dy$. (B.Sc. 44 M)

5. $(x^2-y^2)\dfrac{dy}{dx}=2xy$ given that $y=1$ when $x=1$.

6. $\dfrac{dy}{dx}=\dfrac{y^3+3x^2y}{x^3+3xy^2}$. (B.Sc. 52 M)

7. $x\,dx+y\,dy=2\,(x\,dy-y\,dx)$. (B.Sc. 49 M)

8. $x\dfrac{dy}{dx}+\dfrac{y}{x}=y$.

9. $\dfrac{dy}{dx}=-\dfrac{x^2+3y^2}{3x^2+y^2}$. (B.Sc. 53 M)

10. $(x^2-2xy-y^2)\,dx=(x+y)^2\,dy$. (B.Sc. 52)

11. $x\,(y-x)\,dy=(y+x)\,y\,dx$. (B.E. 48)

12. $(xy-2y^2)\,dx-(x^2-3xy)\,dy=0$. (B.Sc. 57 M)

13. $\dfrac{dy}{dx}=\dfrac{y+\sqrt{x^2+y^2}}{x}$. (B.Sc. Comp. Math. 60)

§ 69·3. TYPE C :—**Non-homogeneous equations of the first degree in x and y.**

Consider $(ax+by+c)\dfrac{dy}{dx}=Ax+By+C$, where a, b, c, A, B, C are constants.

Put $x=X+h$ and $y=Y+k$.

The equation becomes

$$(aX+bY+ah+bk+c)\frac{dY}{dX}=AX+BY+Ah+Bk+C.$$

If h, k be chosen to satisfy

$$ah+bk+c=0 \qquad (i)$$

$$\text{and } Ah+Bk+C=0 \qquad (ii)$$

the above equation reduces to

$$(aX+bY)\frac{dY}{dX}=AX+BY.$$

This is homogeneous in X and Y and can be solved by putting $Y = vX$.

Note 1. The above solution succeeds only if h and k can be solved from (*i*) and (*ii*), *i.e.*, if $\frac{a}{A} \neq \frac{b}{B}$. If, however, $\frac{a}{A} = \frac{b}{B}$ and $\frac{c}{C}$ be different from each of these fractions, h and k cannot be obtained from (*i*) and (*ii*) ; hence the following method is adopted :

Put $\frac{a}{A} = \frac{b}{B} = \frac{1}{m}$.

The given equation is

$$(ax + by + c)\frac{dy}{dx} = [m\,(ax + by) + C].$$

Putting $ax + by = v$, we have

$$\frac{dv}{dx} = a + b\,\frac{mv + C}{v + c}.$$

The variables separate ; hence the solution is

$$\frac{v}{a + bm} + \frac{b\,(mc - C)}{(a + bm)^2}\log\{v\,(a + bm) + ac + bC\} = x + C_1,$$

where C_1 is an arbitrary constant.

2. Suppose further that $\frac{a}{A} = \frac{b}{B} = \frac{c}{C} = \frac{1}{m}$, the equation is $\frac{dy}{dx} = m$.

$\therefore\ y = mx + c_1$ where c_1 is an arbitrary constant.

Examples.

Ex. 1. Solve $\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$. (B.Sc. 52 M)

Put $x = X + h$; $y = Y + k$.

The equation becomes

$$\frac{dY}{dX} = \frac{X + 2Y}{Y + 2X}, \qquad (1)$$

where h and k are chosen to satisfy $h + 2k - 3 = 0$ and $2h + k - 3 = 0$.

$\therefore\ h = 1$ and $k = 1$.

Hence $x = X + 1$ and $y = Y + 1$.

To solve (1), put $Y = vX$; (1) reduces to

$$\frac{dv\,(2 + v)}{1 - v^2} = \frac{dX}{X}.$$

i.e., $\left(\frac{1}{2}\frac{1}{1+v}+\frac{3}{2}\frac{1}{1-v}\right)dv=\frac{dX}{X}$.

Integrating, $1+v=c^2X^2(1-v)^3$.

Hence $X+Y=c^2(X-Y)^3$.

$\therefore\ x+y-2=c^2(x-y)^3$.

Ex. 2. Solve $(2x-4y+3)\frac{dy}{dx}+(x-2y+1)=0$. (B.E. 49)

(Here $\frac{a}{A}=\frac{b}{B}$. *Vide* Note 1.)

Put $x-2y=v$; the equation becomes

$$(4v+5)\,dx-(2v+3)\,dv=0,$$

$$dx-\frac{2v+3}{4v+5}dv=0,$$

i.e., $dx=\left(\frac{1}{2}+\frac{1}{2}\frac{1}{4v+5}\right)dv$.

Integrating, $\frac{v}{2}+\frac{1}{8}\log(4v+5)=x+c$.

The solution is $\log\{4(x-2y)+5\}=4(x+2y)+c_1$.

Ex. 3. Solve $(2x-y+5)\,dy=(x-2y+3)\,dx$. (B.Sc. Anc. 59)

Regrouping, the equation may be written as

$$2(x\,dy+y\,dx)+(-y+5)\,dy=(x+3)\,dx.$$

Integrating, $2xy-\frac{y^2}{2}+5y=\frac{x^2}{2}+3x+c$.

Exercises LXXXVI.

Solve the following equations :—

1. $\frac{dy}{dx}+\frac{10x+8y-12}{7x+5y-9}=0$. (B.E. 45)
2. $\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$. (B.Sc. 48 M)
3. $(x+y-1)\,dy=(x+y+1)\,dx$. (B.E. 50)
4. $\frac{dy}{dx}=\frac{x+7y+2}{3x+5y+6}$. (B.Sc. 50 M)
5. $(3y-7x+7)\,dx+(7y-3x+3)\,dy=0$. (B.E. 46)
6. $(x+y+1)\,dx+(3x+4y+4)\,dy=0$. (B.Sc. 49 M)

7. $(2x + 18y - 14)\dfrac{dy}{dx} = 6x + 5y - 7.$ (B.E. 53)

8. $(6x + 2y - 10)\dfrac{dy}{dx} = 2x + 9y - 20.$ (B.Sc. 53 M)

9. $\dfrac{dy}{dx} = \dfrac{2x + y + 6}{y - x - 3}.$ (B.Sc. 55 M)

10. $(12x + 5y - 9)\,dx + (5x + 2y - 4)\,dy = 0.$

(B.Sc. Comp. Math. 59)

[*Hint.* This can be also done by proper grouping.

$$(12x - 9)\,dx + 5\,(y\,dx + x\,dy) + (2y - 4)\,dy = 0.$$

Integrating, $6x^2 - 9x + 5xy + y^2 - 4y = c.$]

§ 69·4. Type D :—Linear equation.

Definition. A differential equation is said to be linear when the dependent variable and its derivatives occur only in the first degree and no products of these occur.

The linear equation of the first order is of the form $\dfrac{dy}{dx} + Py = Q$, where P and Q are functions of x only. (1)

Consider $\dfrac{dy}{dx} + Py = 0$, *i.e.,* $\dfrac{dy}{y} + P\,dx = 0.$

The solution is $ye^{\int P\,dx} = c.$

Differentiating $e^{\int P\,dx}\left(\dfrac{dy}{dx} + Py\right) = 0.$

This shows that $\dfrac{dy}{dx} + Py = 0$ is rendered an *exact* differential equation, *i.e.,* one which can be deduced from its primitive by mere differentiation and no further operation, on multiplying it by the factor $e^{\int P\,dx}$. Such a factor as this is called an *integrating factor*. So an integrating factor is one which changes a differential equation into an exact differential equation.

Now $e^{\int P\,dx}$ is an integrating factor of (1) by the foregoing analysis. Multiplying (1) by $e^{\int P\,dx}$, we have

$$\left(\frac{dy}{dx} + Py\right)e^{\int P\,dx} = Qe^{\int P\,dx}.$$

Integrating, $ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + c.$

This is the solution of (1).

Examples.

Ex. 1. Solve $\dfrac{dy}{dx} + y\cos x = \dfrac{1}{2}\sin 2x.$

Here $P = \cos x$; $Q = \dfrac{1}{2}\sin 2x.$

$\int P\,dx = \sin x$ and $e^{\int P\,dx} = e^{\sin x}.$

The solution is $ye^{\sin x} = \int \frac{1}{2}\sin 2x\, e^{\sin x}\,dx + c$

$$= \int e^{\sin x}\sin x\cos x\,dx + c$$

$$= \int e^{z}\,z\,dz + c, \text{ on putting } z = \sin x;$$

$$= e^{z}(z-1) + c$$

$$= e^{\sin x}(\sin x - 1) + c.$$

Ex. 2. Solve $x\dfrac{dy}{dx} + y\log x = e^{x}\,x^{1-1/2\log x}.$ (B.A. 44 M)

Reducing the equation to the standard form,

$$\frac{dy}{dx} + \frac{y}{x}\log x = e^{x}\,x^{-1/2\log x}.$$

Here $P = \dfrac{\log x}{x}$; $Q = e^{x}\,x^{-1/2\log x}.$

$$\int P\,dx = \int \frac{\log x}{x}\,dx = \frac{(\log x)^2}{2}.$$

$$\therefore\ e^{\int P\,dx} = e^{\frac{(\log x)^2}{2}} = \left(e^{\log x}\right)^{\frac{\log x}{2}} = x^{\frac{\log x}{2}}.$$

The solution is $y\,x^{\frac{\log x}{2}} = \int e^{x}\,x^{-1/2\log x}\,.\,x^{1/2\log x}\,dx + c$

$$= \int e^{x}\,dx + c = e^{x} + c.$$

Ex. 3. Solve $(1-x^2)\dfrac{dy}{dx} + 2xy = x\sqrt{1-x^2}$ given that $y=0$ when $x=0$.

Here $P = \dfrac{2x}{1-x^2}$; $Q = \dfrac{x}{\sqrt{1-x^2}}.$

$$\int P\,dx = \int \frac{2x\,dx}{1-x^2} = -\log(1-x^2) = \log(1-x^2)^{-1}$$

$$e^{\int P\,dx} = e^{\log(1-x^2)^{-1}} = (1-x^2)^{-1}.$$

The solution is $\dfrac{y}{1-x^2} = \displaystyle\int \dfrac{x\,dx}{(1-x^2)^{3/2}} + c$

$= \displaystyle\int \dfrac{\sin\theta\,d\theta}{\cos^2\theta} + c$ on putting $x = \sin\theta$

$= \int \tan\theta \sec\theta\,d\theta + c$

$= \sec\theta + c = \dfrac{1}{\sqrt{1-x^2}} + c.$

When $x = 0$, $y = 0$. $\therefore$ $c = -1$.

Hence $\dfrac{y}{1-x^2} = \dfrac{1}{\sqrt{1-x^2}} - 1.$

Exercises LXXXVII.

1. $\dfrac{dy}{dx} + y\tan x = \cos^3 x.$ (B.E. 48)

2. $\dfrac{dy}{dx} - \dfrac{xy}{1-x^2} = \dfrac{1}{1-x^2}.$ (B.E. 53)

3. $(1+x^2)\dfrac{dy}{dx} + y = e^{\tan^{-1}x}.$ (B.Sc. 52 M)

4. $\dfrac{dy}{dx} + \dfrac{3x^2y}{1+x^3} = \dfrac{\sin^2 x}{1+x^3}.$

5. $\dfrac{dy}{dx} - \dfrac{2y}{x} = \dfrac{5x^3}{(2+x)(3-2x)}.$ (B.E. 51)

6. $\dfrac{dy}{dx} + \dfrac{2x}{1+x^2}y = \dfrac{1}{(1+x^2)^2}$ given that $y = 0$ when $x = 1$. (B.Sc. 52 M)

7. $\cos^2 x\dfrac{dy}{dx} + y = \tan x.$ (B.Sc. 48 M)

8. $(x^2 - a^2)\dfrac{dy}{dx} + xy = (x+a)\sqrt{x^2-a^2}.$ (B.A. 48)

9. $\dfrac{dy}{dx} + \dfrac{3x^2y}{1+x^3} = \dfrac{1+x^2}{1+x^3}.$ (B.A. 47 M)

10. $\dfrac{dy}{dx} - y\tan x = e^x \sec x.$ (B.A. 42 M)

11. $x(1-x^2)\dfrac{dy}{dx} + (2x^2-1)y = ax^3.$

12. $(x^3+1)\dfrac{dy}{dx} + 2xy = 4x^2.$ (B.A. 51 M)

13. $x(x-1)\dfrac{dy}{dx} - (x-2)y = x^3(2x-1).$ (B.Sc. 50 M)

14. $\frac{dy}{dx}+\frac{3y}{x}=\frac{1}{x^2}$ given that $y=2$ when $x=1$.

15. $\frac{dy}{dx}+\frac{xy}{1+x^2}=\frac{1}{x(1+x^2)}$. (B.A. 48 M)

16. $\frac{dy}{dx}+y\cot x=4x\operatorname{cosec} x$ given that $y=0$ when $x=\pi/2$. (B.A. 38 M)

17. $2x\frac{dy}{dx}+y=2x^3$. (B.Sc. 53 T.U.)

18. $\frac{dy}{dx}+y\cos x=(1+\sin^2 x)\sin 2x$. (B.Sc. Sub. 44)

19. $\frac{dy}{dx}+\frac{y}{x}=xe^x$. (B.A. 45 M)

20. $\frac{dy}{dx}+y(1+x^2)=1+x^2$. (B.Sc. Sub. 42)

21. $x(x-2)\frac{dy}{dx}-2(x-1)y=x^3(x-2)$ given that $y=9$ when $x=3$. (B.Sc. Sub. 40)

22. $\frac{dy}{dx}+2xy=e^{-x^2}x$.

23. The differential equation of an electric current containing a resistance R and a capacity C in series with an electromotive force E is $R\frac{di}{dt}+\frac{i}{c}=\frac{dE}{dt}$. Solve the equation if $E=E_0\cos pt$ (E_0 and p being constants) and $i=0$ at $t=0$.

24. The gradient of a curve which passes through the point $(4, 0)$ is defined by the equation $\frac{dy}{dx}-\frac{y}{x}+\frac{5x}{(x+2)(x-3)}=0$. Find the equation of the curve and find the value of y when $x=5$. (B.Sc. Sub. 45)

25. Solve $\frac{dy}{dx}+\frac{y}{x\log x}=\frac{\sin 2x}{\log x}$. (B.Sc. 46 M)

26. $y''+y'=1$. (*Hint.* Put $y'=z$.) (B.A. 41 M)

27. $y''-y'\tan x=\sin x$.

28. $x\cos x\frac{dy}{dx}+y(x\sin x+\cos x)=1$. (B.A. 54 M)

29. $(x+1)\frac{dy}{dx}-xy=e^x(x+1)^{n+1}$.

30. $\dfrac{dy}{dx}+2y\cot x=3x^2\operatorname{cosec}^2 x.$ (B.Sc. 44)

31. $\dfrac{dy}{dx}+2xy=2x(1+x^2).$ (B.A. 50)

32. $\dfrac{dy}{dx}+2y\tan x=\sin x$ and show that if $y=0$ when $x=\pi/3$, the maximum value of y is 1/8. (B.Sc. 51)

33. $\dfrac{dy}{dx}+(y-1)\cos x=e^{-\sin x}\cot^2 x.$ (B.Sc. Sub. 55)

34. $\dfrac{dy}{dx}+2y\tan x=\sin x.$ (B.Sc. 55 M)

35. $1+(x\tan y-\sec y)\dfrac{dy}{dx}=0.$ (B.Sc. 53 M)

36. $(x+2y^3)\dfrac{dy}{dx}=y.$ (B.Sc. Comp. Math. 59)

[*Hint.* This is linear in x.]

37. $x\dfrac{dy}{dx}-y=\log(x+1).$ (T.U. 55)

38. $\dfrac{dy}{dx}=\dfrac{y}{x}(\log y\log x+1).$ (B.Sc. 53 M)

39. $(1-x^2)\dfrac{dy}{dx}+2xy=x(1-x^2)^{1/2}.$ (B.A. 57)

40. $(1+x^2)y'+2xy=4x^2.$ (B.Sc. Anc. 59)

41. $x^2(x-1)\dfrac{dy}{dx}+x(x^2+1)y=x^2-1.$ (B.Sc. 60)

§ 69·5. Bernoulli's equation.

Consider $\dfrac{dy}{dx}+Py=Qy^n$, where P and Q are functions of x only. This can be reduced to the linear form.

Dividing by y^n, $y^{-n}\dfrac{dy}{dx}+Py^{1-n}=Q$.

Put $z=y^{1-n}$; this equation reduces to

$$\frac{dz}{dx}+Pz(1-n)=Q(1-n).$$

This, being linear in z, can be integrated by the method of § 69·4 and hence y can be got.

Examples.

Ex. 1. Solve $\dfrac{dy}{dx} - y \tan x = \dfrac{\sin x \cos^2 x}{y^2}$.

Multiplying by y^2, $y^2 \dfrac{dy}{dx} - y^3 \tan x = \sin x \cos^2 x$.

Put $z = y^3$; $\dfrac{dz}{dx} - 3z \tan x = 3 \sin x \cos^2 x$.

Here $P = -3 \tan x$; $Q = 3 \sin x \cos^2 x$.

$\int P\, dx = 3 \log \cos x = \log \cos^3 x$.

$\therefore\ e^{\int P\, dx} = \cos^3 x$.

Hence $z \cos^3 x = 3 \int \sin x \cos^5 x\, dx + c = -\dfrac{\cos^6 x}{2} + c$

$$\therefore\ y^3 \cos^3 x = -\frac{\cos^6 x}{2} + c.$$

Ex. 2. Solve $(x+1)\dfrac{dy}{dx} + 1 = 2e^{-y}$. (B.Sc. Sub. 51)

Multiplying by e^y and putting $z = e^y$, $(x+1)\dfrac{dz}{dx} + z = 2$.

$$\therefore\ \frac{dz}{dx} + \frac{z}{x+1} = \frac{2}{x+1}.$$

Here $P = \dfrac{1}{x+1}$; $Q = \dfrac{2}{x+1}$.

$e^{\int P\, dx} = e^{\log (x+1)} = x + 1$.

$\therefore\ z(x+1) = \displaystyle\int \frac{2}{x+1}(x+1)\, dx + c = 2x + c.$

$\therefore\ e^y (x+1) = 2x + c$.

Exercises LXXXVIII.

Solve the equations : $\left(y' \text{ stands for } \dfrac{dy}{dx} \right)$

1. $y' + y \cos x = y^n \sin 2x$. (B.E. 47 & B.Sc. 61)
2. $y' - \dfrac{2}{x} y = \dfrac{y^3}{x^3}$. (B.Sc. 48 M)
3. $y' = x^2 y^3 - xy$. (B.Sc. Sub. 51)
4. $(1 - x^2) y' - xy = x^2 y^2$. (B.Sc. 44 M)
5. $y' + x^3 y = e^x y^4$. (B.E. 49)
6. $xy' + y = y^2 \log x$.

7. $3x(1-x^2)y^2y'+(2x^2-1)y^3=ax^3.$ (B.Sc. 51 M)

8. $2y'(1+x^2)-y+y^3=0.$ (B.Sc. 45 M)

9. $(x^3e^x-2my^2)\,dx+2m\,xy\,dy=0.$ (B.E. 51)

10. $xy'+y=x^3y^6.$ (B.E. 53)

11. $y'=1-x(y-x)-x^3(y-x)^2.$ (B.Sc. 53 T.U.)

12. $2y^3\dfrac{dy}{dx}+\dfrac{y^4}{x}=e^x.$ (B.Sc. 54 M)

13. $y'+y/x=y^n/x^n.$ (B.Sc. 38)

14. $y'/y^2+1/xy=1.$ (B.Sc. 47)

15. $\dfrac{dy}{dx}-\dfrac{\tan y}{1+x}=(1+x)e^x\sec y.$ (B.Sc. 48 M)

16. $\dfrac{dy}{dx}2y\cos y^2-\dfrac{2}{x+1}\sin y^2=(x+1)^3.$ (B.E. 55)

17. $x^2-y^2+2xy\dfrac{dy}{dx}=0.$ (B.Sc. Sub. 57)

18. $\dfrac{dx}{dy}-\dfrac{2}{3}xy=x^4y^3.$ (B.A. 59)

19. $\dfrac{dy}{dx}+\dfrac{1}{x}=\dfrac{e^y}{x^2}.$ (B.Sc. 59)

20. $\dfrac{dy}{dx}+\dfrac{x}{1-x^2}y=x\sqrt{y}.$ (B.Sc. Anc. 59)

21. $2\dfrac{dy}{dx}=\dfrac{y}{x}+\dfrac{y^2}{x^2}.$ (B.Sc. Comp. Math. 59)

22. $a\dfrac{dy}{dx}+y=\dfrac{x}{y^{n-1}}.$ (B.Sc. Anc. 60)

§ 70. Linear Equations of the second order with constant coefficients.

A typical linear equation of the second order with constant coefficients is

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+c=X, \tag{1}$$

where a, b, c are constants and X is a function of x.

Let us consider (1) without the second member,

$$i.e.,\quad a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+c=0. \tag{2}$$

The solution of this equation (2) is called the *complementary function* of (1).

To solve (2), assume as a trial solution $y = e^{mx}$ for some value of m. Now $\frac{dy}{dx} = me^{mx}$ and $\frac{d^2y}{dx^2} = m^2e^{mx}$. Substituting these values in (2), we get $e^{mx}(am^2 + bm + c) = 0$.

Hence m satisfies $am^2 + bm + c = 0$. (3)

This equation in m is called the *auxiliary* equation.

Three cases can arise in the solution of the auxiliary equation.

Case i. Let the auxiliary equation (3) have two real and distinct roots m_1 and m_2.

$\therefore\ y = e^{m_1x}$ and $y = e^{m_2x}$ are solutions of (2).

Hence Ae^{m_1x} and Be^{m_2x} are also solutions of (2), where A and B are arbitrary constants. Thus $y = Ae^{m_1x} + Be^{m_2x}$ is the most general solution of (2), as the number of constants occurring in this solution is two, equal to the order of the differential equation.

Case ii. Let the auxiliary equation (3) have two roots equal and real.

Let $m_2 = m_1$. The solution $y = Ae^{m_1x} + Be^{m_2x}$ becomes $(A + B)\,e^{m_1x} = c\,e^{m_1x}$ (4) where c is a single arbitrary constant equal to $A + B$. Thus the number of constants is reduced to one which is one short of the order of the differential equation (2) and therefore (4) ceases to represent the general solution. Hence we proceed as follows :—

Let us put $m_2 = m_1 + \epsilon$ and allow ϵ to tend to zero.

The solution is $y = Ae^{m_1x} + Be^{(m_1+\epsilon)x}$

$$= e^{m_1x}(A + Be^{\epsilon x})$$

$$= e^{m_1x}\left[A + B\left(1 + \epsilon x + \frac{\epsilon^2x^2}{2\,!} + \dots\right)\right]$$

by the exponential theorem

$= e^{m_1x}\ (A + B + \epsilon Bx)$, the other terms tending to zero as $\epsilon \to 0$.

We can choose B sufficiently big so as to make ϵB finite as $\epsilon \to 0$ and A large with opposite sign to B so that $A + B$ is finite. If $A + B = C$ and $\epsilon B = D$, the solution corresponding to two equal roots m_1 is $e^{m_1x}(C + Dx)$.

Aliter.

Let the auxiliary equation $am^2 + bm + c = 0$ have both roots equal to m_1.

$\therefore \quad am^2 + bm + c \equiv a(m - m_1)^2.$

If D stands for $\frac{d}{dx}$, (*vide* § 71) and D^2 for $\frac{d^2}{dx^2}$, the given differential equation is

$$(aD^2 + bD + c)\,y = a\,(D - m_1)^2\,y = 0$$
$$i.e., \quad (D - m_1)\,(D - m_1)\,y = 0 \text{ as } a \neq 0.$$

Put $(D - m_1)\,y = z.$ (2)

Then $(D - m_1)\,z = 0$, *i.e.*, $\frac{dz}{dx} - m_1 z = 0.$

$\therefore \quad z = Ae^{m_1x}.$

Hence (1) becomes $\frac{dy}{dx} - m_1 y = Ae^{m_1x}.$

Solving this linear equation $y\,e^{-m_1x} = \int A\,dx + B = Ax + B$, *i.e.*, $y = (Ax + B)\,e^{m_1x}.$

Case iii. Let the auxiliary equation have imaginary roots.

As imaginary roots occur in pairs, let $m_1 = \alpha + i\beta$, where δ and β are real ; then $m_2 = \alpha - i\beta.$

The solution is
$$\begin{aligned} y &= Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} \\ &= e^{\alpha x}\,[Ae^{i\beta x} + Be^{-i\beta x}] \\ &= e^{\alpha x}\,[A(\cos\beta x + i\sin\beta x) \\ &\qquad + B(\cos\beta x - i\sin\beta x)] \end{aligned}$$

By Euler's formula

$$= e^{\alpha x}\,(C\cos\beta x + D\sin\beta x),$$

where C and D are arbitrary constants.

Examples.

Ex. 1. Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0.$

The auxiliary equation is $m^2 - 5m + 4 = 0.$

Solving, $m = 1$ and 4.

$\therefore \quad y = Ae^{x} + Be^{4x}.$

Ex. 2. Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0.$

The auxiliary equation is $m^2 + 2m + 1 = 0$, *i.e.*, $(m + 1)^2 = 0$, $m = -1$ twice.

$\therefore \quad y = e^{-x}\,(A + Bx).$

Ex. 3. Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 5y = 0$.

The auxiliary equation is $m^2 - 3m + 5 = 0$.

Solving, $m = \frac{3 \pm \sqrt{11}\, i}{2}$.

$$\therefore \quad y = e^{\frac{3x}{2}} \left\{ A \cos\left(\frac{\sqrt{11}}{2} x\right) + B \sin\left(\frac{\sqrt{11}}{2} x\right) \right\}.$$

Exercises LXXXIX.

Solve the following equations :—

1. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$.
2. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 4y = 0$.
3. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$.
4. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 8y = 0$.
5. $2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 4y = 0$.
6. $\frac{d^2y}{dx^2} + 16y = 0$.
7. $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$.
8. $(D^2 - 2D - 15)\, y = 0$ given that $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 2$ when $x = 0$. (B.Sc. Anc. 60)

§ 71. The operators D and D^{-1}.

Let D stand for the operator $\frac{d}{dx}$ and D^2 for $\frac{d^2}{dx^2}$.

This symbol D satisfies the commutative, associative and distributive laws ; for

$$(D^m + D^n)\, u = (D^n + D^m)\, u = D^n u + D^m u$$
$$D^m \,.\, D^n u = D^n \,.\, D^m u = D^{m+n} u$$

and $D\,(u + v) = D\,(v + u)$.

We can define the inverse operator D^{-1} as one such that when it operates on any function of x and subsequently the operation by D is performed, the function is left unaltered. Thus D^{-1} represents an integration.

We shall define the operator $\dfrac{1}{f(D)}$ as the inverse of the operator $f(D)$, *i.e.*, $\dfrac{1}{f(D)}X$ is that function of x, which, when operated upon by $f(D)$, yields X.

We shall assume that the order of the operators $f(D)$ and $\dfrac{1}{f(D)}$ can be interchanged.

Then $f(D)\left\{\dfrac{1}{f(D)}X\right\} = \dfrac{1}{f(D)}\{f(D)X\} = X.$

§ 72. Particular integral.

Consider equation (1) of § 70 which can be written symbolically as $(aD^2 + bD + c)y = X$ (1)
or shortly $f(D)y = X$ where $f(D) \equiv aD^2 + bD + c$.

Let $y = u$ be a particular solution of this linear equation. Let y be the complementary function of (1) (*vide* § 70).

Then $y = Y + u$ is the general solution of (1).

u is called the *particular integral* of (1).

In symbolic form, it is written as $\dfrac{1}{f(D)}X$

i.e., $P.I. = \dfrac{1}{aD^2 + bD + c}X.$

§ 73. Special Methods of finding P.I.

(a) Let X be of the form $e^{\alpha x}$.

$D\,e^{\alpha x} = \alpha\,e^{\alpha x}.$

More generally, $D^n e^{\alpha x} = \alpha^n e^{\alpha x}.$

$\therefore\ f(D)e^{\alpha x} = f(\alpha)e^{\alpha x}$ as $f(D)$ is a quadratic in D in our case.

Operating on both sides by $\dfrac{1}{f(D)}$,

$$e^{\alpha x} = f(\alpha)\frac{1}{f(D)}e^{\alpha x}.$$

If $f(\alpha) \neq 0$, $\dfrac{1}{f(D)}e^{\alpha x} = \dfrac{1}{f(\alpha)}e^{\alpha x}.$

Case i. Hence the rule is :—

In $\dfrac{1}{f(D)}e^{\alpha x}$, replace D by α if $f(\alpha) \neq 0$

i.e., α is not a root of the auxiliary equation $f(m) = 0$.

Case ii. If $f(a) = 0$, a satisfies the auxiliary equation $f(m) = 0$. Then we proceed as follows.

(*i*) Let the auxiliary equation have two distinct roots m_1 and m_2 and let $a = m_1$.

Then $f(m) \equiv a(m - m_1)(m - m_2)$

$$= a(m - a)(m - m_2).$$

$$P.I. = \frac{1}{a(D-a)(D-m_2)} e^{ax}$$

$$= \frac{1}{a(D-a)} \frac{1}{(a-m_2)} e^{ax} \text{ by Case i above.}$$

To find $\frac{1}{D-a} e^{ax}$, let us put $z = \frac{1}{D-a} e^{ax}$.

Operating on both sides by $D - a$,

$$\frac{dz}{dx} - az = e^{ax}.$$

This is a linear equation of the first order ; hence

$$ze^{-ax} = \int e^{-ax} e^{ax}\, dx = x.$$

[It must be noted that no constant of integration is added as we are evaluating only a particular integral.

If the constant be added, there will occur in the general solution 3 constants as there are already 2 in the *C.F.* and thus one constant will be too many.]

$\therefore \quad z = x e^{ax}.$

Hence $P.I. = \frac{1}{a(a-m_2)} xe^{ax}$.

(*ii*) Let the auxiliary equation have two equal roots, each equal to a, *i.e.*, $m_2 = m_1 = a$.

$\therefore \; f(m) = a(m-a)^2.$

$$P.I. = \frac{1}{a(D-a)^2} e^{ax} = \frac{1}{a(D-a)(D-a)} e^{ax}$$

$$= \frac{1}{a} \frac{x e^{ax}}{D-a}.$$

If $z = \frac{1}{D-a} x e^{ax}$, $\frac{dz}{dx} - az = x e^{ax}$.

Solving, $z e^{-ax} = \int x\, dx = \frac{x^2}{2}$ (no constant is added).

$$\therefore \quad z = \frac{x^2}{2} e^{ax}.$$

$$\therefore \quad P.I. = \frac{1}{a} \frac{x^2}{2} e^{ax}.$$

Examples.

Ex. 1. Solve $(D^2 + 5D + 6) y = e^x$. (B.Sc. 48 M)

To find the $C.F.$, solve $(D^2 + 5D + 6) y = 0$.

The auxiliary equation is $m^2 + 5m + 6 = 0$.

Solving, $m = -2$ and -3.

$C.F. = Ae^{-2x} + Be^{-3x}$.

$$P.I. = \frac{1}{D^2 + 5D + 6} e^x$$

$$= \frac{1}{12} e^x \text{ on replacing } D \text{ by } 1 \text{ ; § 73, Case i.}$$

$$y = Ae^{-2x} + Be^{-3x} + \frac{e^x}{12}.$$

Ex. 2. Solve $(3D^2 + D - 14) y = 13e^{2x}$. (B.Sc. 50 M)

To find the $C.F.$, solve $(3D^2 + D - 14) y = 0$.

The auxiliary equation is $3m^2 + m - 14 = 0$.

Solving, $m = 2$ and $-7/3$.

$\therefore \quad C.F. = Ae^{2x} + Be^{-7x/3}$.

$$P.I. = \frac{1}{(D-2)(3D+7)} 13e^{2x}$$

$$= \frac{13}{13} \frac{1}{D-2} e^{2x} \text{ by § 73, Case i}$$

$$= \frac{1}{D-2} e^{2x} = xe^{2x} \text{ by § 73, Case ii.}$$

$$\therefore \quad y = Ae^{2x} + Be^{-7x/3} + xe^{2x}.$$

Ex. 3. Solve $(D^2 - 2mD + m^2) y = e^{mx}$. (B.Sc. 52 M)

To find the $C.F.$, solve $(D^2 - 2mD + m^2) y = 0$.

The auxiliary equation is $k^2 - 2mk + m^2 = 0$.

(*Note.* k is used here instead of the usual m as there is already another m.)

i.e., $(k - m)^2 = 0$. $\therefore k = m$ twice.

$C.F. = e^{mx} (A + Bx)$.

$$P.I. = \frac{1}{(D-m)^2} e^{mx} = \frac{x^2}{2} e^{mx} \text{ by § 73, Case ii.}$$

$$\therefore y = e^{mx} \left(A + Bx + \frac{x^2}{2} \right).$$

Exercises XC. Solve the following equations :—

1. $(D^2-5D+6)\,y=e^{4x}$. (B.A. 47 M)
2. $(D^2-6D+13)\,y=5e^{2x}$. (B.Sc. 49 M)
3. $(D^2+4D+6)\,y=5e^{-2x}$. (B.Sc. Sub. 45)
4. $(D^2+2D+1)\,y=2e^{3x}$. (B.A. 52 M)
5. $(D^2-3D+2)\,y=e^{3x}$ which shall vanish for $x=0$ and for $x=\log 2$.
6. $(D^2-13D+12)\,y=e^{-2x}+5e^{x}$.
7. $(D^2-12D+16)\,y=(e^{x}+e^{-2x})^2$.
8. $(D^2+6D+8)\,y=e^{-2x}$. (B.A. 43 M)
9. $(D^2-6D+9)\,y=2e^{-x}+e^{3x}$.
10. $(D^2+31D+240)\,y=272e^{-x}$. (B.A. 36 M)
11. $(4D^2+16D+15)\,y=4e^{-3x/2}$ given that $y=3$ and $Dy=-5{\cdot}5$ when $x=0$. (B.Sc. Sub. 45)
12. $(D^2+3D+2)\,y=e^{4x}$ given $y=0$ when $x=0$ and $x=1$. (B.Sc. Sub. 42)
13. $(D^2+5D+7)\,y=e^{-x/2}+5$.
14. $(D^2+2pD+p^2+q^2)\,y=e^{ax}$ where p and q are constants. (B.A. 47 M)
15. $(D^2-2D+1)\,y=e^{x}+e^{2x}$. (B.Sc. Sub. 52)
16. $(D^2+D-6)\,y=e^{3x}+e^{-3x}$. (B.Sc. 53)
17. $(D^2-3D+2)\,y=2e^{3x}$, such that $y=0$ when $x=0$ and $x=\log 2$. (B.Sc. 55 M)
18. $(D^2-a^2)\,y=e^{ax}+e^{ax}$. (B.Sc. Anc. 60)
19. $(D^2+4D+8)\,y=(1+e^{x})^2$. (B.Sc. 60)

(b) Let X be of the form cos ax **or sin** ax, where a is a *constant*.

$D\sin ax=a\cos ax,$

$D^2\sin ax=-a^2\sin ax.$

$\therefore\ \phi(D^2)\sin ax=\phi(-a^2)\sin ax$, as $\phi(D^2)$ is a rational integral function of D^2.

Operating on both sides by $\dfrac{1}{\phi(D^2)}$,

$$\sin ax=\frac{\phi(-a^2)}{\phi(D^2)}\sin ax.$$

Case i. If $\phi(-a^2)\neq 0$,

$$\frac{1}{\phi(D^2)}\sin ax=\frac{1}{\phi(-a^2)}\sin ax.$$

Hence the rule is :—

Replace D^2 by $-a^2$, provided $\phi(-a^2) \neq 0$.

The same rule applies if $\sin ax$ be replaced by $\cos ax$

i.e., $\dfrac{1}{\phi(D^2)} \cos ax = \dfrac{1}{\phi(-a^2)} \cos ax.$

Case ii. If $\phi(-a^2) = 0$, $D^2 + a^2$ is a factor of $\phi(D^2)$. To evaluate $\dfrac{1}{D^2 + a^2} \sin ax$, the above rule fails. Hence the following procedure is adopted.

$$\frac{1}{D^2 + a^2} \sin ax = \frac{1}{D^2 + a^2} \times \text{Imaginary Part } e^{aix} \text{ as}$$

$$e^{iax} = \cos ax + i \sin ax \text{ by Euler's formula ;}$$

$$= \text{Imaginary Part of } \frac{1}{D^2 + a^2} e^{iax}$$

$$= \quad ,, \quad \frac{1}{(D - ai)(D + ai)} e^{iax}$$

$$= \quad ,, \quad \frac{1}{(D - ai)\, 2ai} e^{iax} \text{ by § 73 (a)}$$

$$= \quad ,, \quad \frac{xe^{iax}}{2ai} \text{ by § 73 (a)}$$

$$= \quad ,, \quad -\frac{xi}{2a} (\cos ax + i \sin ax)$$

$$= -\frac{x \cos ax}{2a}.$$

Examples.

Ex. 1. Solve $(D^2 - 3D + 2)\, y = \sin 3x$. (B.Sc. Sub. 46)

To find the *C.F.*, solve $(D^2 - 3D + 2)\, y = 0$.

The auxiliary equation is $m^2 - 3m + 2 = 0$.

Solving, $m = 1$ and 2.

$C.F. = Ae^x + Be^{2x}$.

$$P.I. = \frac{1}{D^2 - 3D + 2} \sin 3x$$

$$= \frac{1}{-9 - 3D + 2} \sin 3x \text{ on replacing } D^2 \text{ by } -9 \text{ by § 73 (b)}$$

$$= \frac{-1}{3D + 7} \sin 3x.$$

In order to apply the above rule, we must aim at getting D^2 terms only in the denominator ; hence we write

$$\frac{1}{3D+7} \text{ as } \frac{3D-7}{(3D-7)(3D+7)} = \frac{3D-7}{9D^2-49}$$

and proceed.

$$P.I. = \frac{-(3D-7)}{9D^2-49} \sin 3x$$

$$= \frac{-3\frac{d}{dx}(\sin 3x) + 7 \sin 3x}{9(-9)-49} \text{ by § 73 } (b)$$

$$= \frac{-9 \cos 3x + 7 \sin 3x}{-130}.$$

$$y = C.F. + P.I.$$

Ex. 2. Show that the solution of the differential equation $\frac{d^2y}{dt^2} + 4y = A \sin pt$ which is such that $y = 0$ and $\frac{dy}{dt} = 0$ when $t = 0$ is $y = A\frac{(\sin pt - \frac{1}{2} p \sin 2t)}{4-p^2}$ if $p \neq 2$.

If $p = 2$, show that $y = \frac{A(\sin 2t - 2t \cos 2t)}{8}$.

(B.Sc. 51 M)

Let D stand for $\frac{d}{dt}$ here.

To find the $C.F.$, solve $(D^2+4)y = 0$.

The auxiliary equation is $m^2 + 4 = 0$. Solving, $m = \pm 2i$.

$\therefore$ $C.F. = \lambda \cos 2t + \mu \sin 2t$, where λ and μ are arbitrary constants.

(Note that the independent variable is t.)

$$P.I. = \frac{1}{D^2+4} A \sin pt$$

$$= \frac{1}{-p^2+4} A \sin pt \text{ if } p^2 \neq 4 \text{ by § 73 } (b)$$

$$\therefore y = \lambda \cos 2t + \mu \sin 2t + \frac{A}{4-p^2} \sin pt.$$

To determine the values of λ and μ, we note that when $t = 0$, $y = 0$ and $\frac{dy}{dt} = 0$.

$\therefore \quad 0 = \lambda.$ (1)

$$\frac{dy}{dt} = -2\lambda \sin 2t + 2\mu \cos 2t + \frac{Ap}{4-p^2} \cos pt.$$

$$\therefore\ 0 = 2\mu + \frac{Ap}{4-p^2}.$$

$$\mu = -\frac{Ap}{2(4-p)^2}. \quad (2)$$

Hence, $y = \dfrac{A(\sin pt - \frac{1}{2} p \sin 2t)}{4-p^2}.$

If $p = 2$, $P.I. = \dfrac{1}{D^2+4} A \sin 2t$

$= \text{Imaginary part of } \dfrac{A}{D^2+4} e^{i\cdot 2t}$

$= \quad \text{,,} \quad \dfrac{A}{(D+2i)(D-2i)} e^{i\cdot 2t}$

$= \quad \text{,,} \quad \dfrac{A}{4i} te^{2it}$

$= \quad \text{,,} \quad \dfrac{-At}{4} i (\cos 2t + i \sin 2t)$

$= \dfrac{-At \cos 2t}{4}.$

$$y = \lambda \cos 2t + \mu \sin 2t - \frac{At}{4} \cos 2t.$$

When $t = 0$, $y = 0$. $\therefore \quad \lambda = 0.$

$$\frac{dy}{dt} = -2\lambda \sin 2t + 2\mu \cos 2t - \frac{A}{4}(\cos 2t - 2t \sin 2t).$$

$$0 = 2\mu - \frac{A}{4}. \quad \therefore \quad \mu = \frac{A}{8}.$$

$$y = \frac{A(\sin 2t - 2t \cos 2t)}{8}.$$

Ex. 3. Solve $(D^2 - 4D + 3) y = \sin 3x \cos 2x$.

To find the $C.F.$, solve $(D^2 - 4D + 3)y = 0$.

The auxiliary equation is $m^2 - 4m + 3 = 0$.

Solving, $m = 1$ and 3.

$C.F. = Ae^x + Be^{3x}.$

$P.I. = \dfrac{1}{D^2 - 4D + 3} \sin 3x \cos 2x$

$= \dfrac{1}{D^2 - 4D + 3} \dfrac{\sin 5x + \sin x}{2}$

$$= \frac{1}{-25-4D+3}\frac{\sin 5x}{2} + \frac{1}{-1-4D+3}\frac{\sin x}{2}$$

$$= \frac{2D-11}{-4(4D^2-121)}\sin 5x + \frac{1+2D}{4(1-4D^2)}\sin x$$

$$= \frac{10\cos 5x - 11\sin 5x}{884} + \frac{\sin x + 2\cos x}{20}.$$

$y = C.F. + P.I.$

Ex. 4. Solve $(D^2+16)\,y = e^{-3x} + \cos 4x$.

To find the $C.F.$, solve $(D^2+16)\,y = 0$.

The auxiliary equation is $m^2 + 16 = 0$.

Solving, $m = \pm 4i$. $C.F. = A\cos 4x + B\sin 4x$.

$$P.I._1 \text{ corresponding to } e^{-3x} = \frac{1}{D^2+16}e^{-3x}$$

$$= \frac{1}{25}e^{-3x} \text{ by § 73 } (a).$$

$$P.I._2 \text{ corresponding to } \cos 4x = \frac{1}{D^2+16}\cos 4x$$

$$= \frac{1}{D^2+16} \text{ Real part of } e^{4ix}$$

$$= \text{Real part of } \frac{1}{(D+4i)(D-4i)}e^{4ix}$$

$$= \quad ,, \quad \frac{1}{8i}xe^{4ix}$$

$$= \quad ,, \quad -\frac{xi}{8}(\cos 4x + i\sin 4x)$$

$$= \frac{x}{8}\sin 4x.$$

$$\therefore \; y = A\cos 4x + B\sin 4x + \frac{1}{25}e^{-3x} + \frac{x}{8}\sin 4x.$$

Exercises XCI.

Solve the following equations :—

1. $(D^2+4)\,y = \sin 3x$. (B.Sc. 48 M)
2. $(D^2+D+1)\,y = \sin 2x$. (B.Sc. 50 M)
3. $(D^2-8D+9)\,y = 8\cos 5x$. (B.A. 48 M)
4. $(D^2-2D-8)\,y = 4\cos 2x$.
5. $(D^2+n^2)\,x = \cos nt + 3\sin nt$ (n being a constant). (B.A. 46 M)
6. $(D^2-3D+2)\,y = \cos 3x \cos 2x$.

7. $(D^2 - 4)\, y = \sin^2 x$.
8. $(D^2 + 5D - 6)\, y = \sin 4x \sin x$.
9. $(D^2 + 9)\, y = \cos^3 x$.
10. $(D^2 - 4)\, y = \sin^3 x - \cos^2 x$.
11. $(D^2 - D - 2)\, y = 20 \sin 2x + 4e^{3x}$. (B.A. 50 M)
12. $(D^2 - 4D - 5)\, y = e^{2x} + 3 \cos 4x$.
13. $(D^2 + 5D + 6)\, y = e^{-2x} + \sin 4x$. (B.Sc. 51 M)
14. $(D^2 + n^2)\, x = f \cos (nt + a)$. (B.Sc. 52 M)
15. $(D^2 + kD + \mu)\, x = a \cos (nt + a)$.
16. $(D^2 + a^2)\, y = \cos ax$.
17. $(D^2 + 9)\, y = \cos 3x + 3 \sin 2x$.
18. $(D^2 + 9)\, x = 4 \cos (t + \pi/3)$ which is satisfied by $x = 0$ when $t = 0$ and by $x = 2$, when $t = \pi/6$.
19. $(D^2 + 1)\, y = 2 \cos^2 t/2$. (B.A. 45 M)
20. $(D^2 - 9)\, y = \cosh x + \cos x$.
21. $(2D^2 - 7D + 3)\, y = \sin^2 x$.
22. $(D^2 + 4)\, y = e^x + \sin 2x$. (B.A. 52 M)
23. $(D^2 - 4D - 5)\, y = e^{3x} + 4 \cos 3x$. (B.E. 50)
24. $(D^2 + D + 1)\, y = \cos x$. (B.Sc. 44 M)
25. $(3D^2 + D - 14)\, y = 8e^{2x} + \cos 5x$. (B.E. 50)
26. $(D^2 - 6D + 13)\, y = \sin 2x$. (B.A. 55 M)
27. $(D^2 + 25)\, y = -\sin 6x$. (B.Sc. Sub. 55)
28. $(D^2 - 2D + 1)\, y = e^{2x} - \cos 3x$. (B.Sc. Sub. 57)
29. $(4D^2 + 5D)\, y = \sin x$. (B.Sc. 61)
30. $(D^2 + 2n \cos aD + n^2)\, x = a \cos nt$, given that $x = 0$, $Dx = 0$ at $t = 0$ where $D \equiv \dfrac{d}{dt}$. (B.Sc. Anc. 61)

(c) Let X be of the form x^m (a power of x), m being a positive integer.

To evaluate $\dfrac{1}{f(D)} x^m$, raise $f(D)$ to power -1 and expand in ascending powers of D as far as D^m. (The higher powers of D operating on x^m give zero and hence are omitted.) These terms in the expansion of $\{f(D)\}^{-1}$ operating on x^m give the particular integral required.

Examples.

Ex. 1. Solve $(D^2 + D + 1)\, y = x^2$. (B.Sc. Sub. 40)

To find the *C.F.*, solve $(D^2 + D + 1)\, y = 0$.

The auxiliary equation is $m^2 + m + 1 = 0$.

Solving, $m = \dfrac{-1 \pm \sqrt{3}i}{2}$.

$$C.F. = e^{-x/2}\left(A \cos \frac{\sqrt{3}x}{2} + B \sin \frac{\sqrt{3}x}{2}\right).$$

$$P.I. = \frac{1}{1 + D + D^2} x^2$$

$$= (1 + \overline{D + D^2})^{-1} x^2$$

$$= \{ 1 - (D + D^2) + (D + D^2)^2 \} x^2, \text{ the powers of } D \text{ higher than 2 are dropped}$$

$$= (1 - D) x^2 = x^2 - 2x.$$

$$y = C.F. + P.I.$$

Aliter for P.I.

Assume $y = Ax^2 + Bx + C$ for $P.I.$, where A, B, C are constants.

$$\frac{dy}{dx} = 2Ax + B \text{ and } \frac{d^2y}{dx^2} = 2A.$$

Substituting these values in the differential equation,

$$2A + 2Ax + B + Ax^2 + Bx + C = x^2.$$

Equating coefficients of x^2, x and constant term on both sides,

$$A = 1\ ;\ 2A + B = 0 \text{ and } 2A + B + C = 0.$$

$$B = -2 \text{ and } C = 0.$$

Hence $P.I. = x^2 - 2x$.

Ex. 2. Solve $(D^2 + 4D + 5)\, y = e^x + x^3 + \cos 2x$.

To find the $C.F.$, solve $(D^2 + 4D + 5)\, y = 0$.

The auxiliary equation is $m^2 + 4m + 5 = 0$.

Solving, $m = -2 \pm i$.

$C.F. = e^{-2x} (A \cos x + B \sin x)$.

$$P.I._1 \text{ corresponding to } e^x = \frac{1}{D^2 + 4D + 5} e^x$$

$$= \frac{1}{1 + 4 + 5} e^x = \frac{1}{10} e^x.$$

$$P.I._2 \text{ corresponding to } x^3 = \frac{1}{5 + 4D + D^2} x^3$$

$$= \frac{1}{5}\left(1 + \frac{4D + D^2}{5}\right)^{-1} x^3$$

$$=\frac{1}{5}\left[1-\left(\frac{4D+D^2}{5.}\right)+\left(\frac{4D+D^2}{5}\right)^2-\left(\frac{4D+D^2}{5}\right)^3\right]x^3$$

expanding as far as D^3

$$=\frac{1}{5}\left\{1-\frac{4D}{5}+\frac{11D^2}{25}-\frac{24}{125}D^3\right\}x^3$$

$$=\frac{1}{5}\left(x^3-\frac{12x^2}{5}+\frac{66}{25}x-\frac{144}{125}\right).$$

($P.I._2$ can also be found by assuming
$y = Ax^3 + Bx^2 + Cx + D$).

$P.I._3$ corresponding to $\cos 2x = \dfrac{1}{D^2+4D+5}\cos 2x$

$$=\frac{1}{1+4D}\cos 2x \text{ on putting } -4 \text{ for } D^2$$

$$=\frac{1-4D}{1-16D^2}\cos 2x=\frac{\cos 2x+8\sin 2x}{65}$$

$$y = C.F. + P.I._1 + P.I._2 + P.I._3.$$

Exercises XCII.

Solve the following equations :—

1. $(D^2-1)\,y = 2+5x.$
2. $(D-1)^2 y = x.$ (B.Sc. Sub. 43)
3. $(D^2+D+1)\,y = x.$ (B.A. 41 M)
4. $(D^2-5D+6)\,y = x^2-x+2.$
5. $(D^2+3D+2)\,y = \sin x + x^2.$
6. $(D^2+9)\,y = e^{2x}+2x^2.$
7. $(D^2+4)\,y = x+\cos 2x.$ (B.A. 45 M)
8. $(D^2+3D+2)\,y = e^{-x}+x^2+\cos x.$
9. $(D^2+3D-4)\,y = x^2-2x.$ (B.Sc. Sub. 45)
10. $(D^2-2D+1)\,y = x+1.$ (B.Sc. Comp. Math. 60)

(d) X is of the form e^{ax} V, where V is any function of x.

$D\,(e^{ax}V) = ae^{ax}V + e^{ax}DV = e^{ax}(D+a)\,V.$

$D^2\,(e^{ax}V) = D\,\{e^{ax}(D+a)\,V\}$

$= ae^{ax}(D+a)\,V + e^{ax}(D^2+aD)\,V$

$= e^{ax}(D^2+2aD+a^2)\,V = e^{ax}(D+a)^2\,V.$

It follows by induction that $D^n(e^{ax}V) = e^{ax}(D+a)^n\,V.$

$\therefore\ f(D)\,[e^{ax}V] = e^{ax}f(D+a)\,V.$

Operating on both sides by $\dfrac{1}{f(D)}$,

$$e^{ax}V=\frac{1}{f(D)}e^{ax}f(D+a)\,V.$$

If we set $f(D+a)\,V = V_1$, then this result gives

$$e^{ax}\frac{1}{f(D+a)}V_1 = \frac{1}{f(D)}e^{ax}V_1.$$

Hence $\dfrac{1}{f(D)}e^{ax}X = e^{ax}\dfrac{1}{f(D+a)}X.$

Examples.

Ex. 1. Solve $(D^2-4D+3)\,y = e^{-x}\sin x$. (B.Sc. Sub. 35)

To find the $C.F.$, solve $(D^2-4D+3)\,y = 0$.

The auxiliary equation is $m^2-4m+3=0$.

$\therefore\ m = 1$ and 3.

$C.F. = Ae^x + Be^{3x}.$

$$P.I. = \frac{1}{D^2-4D+3}e^{-x}\sin x$$

$$= e^{-x}\frac{1}{(D-1)^2-4(D-1)+3}\sin x \text{ by the above rule}$$

$$= e^{-x}\frac{1}{D^2-6D+8}\sin x$$

$$= e^{-x}\frac{1}{7-6D}\sin x \text{ on putting } -1 \text{ for } D^2$$

$$= e^{-x}\frac{7+6D}{49-36D^2}\sin x$$

$$= e^{-x}\frac{7\sin x+6\cos x}{85}.$$

$y = C.F. + P.I.$

Ex. 2. Solve $(D^2+2D+5)\,y = xe^x$.

To find the $C.F.$, solve $(D^2+2D+5)\,y=0$.

The auxiliary equation is $m^2+2m+5=0$.

Solving, $m = -1 \pm 2i$.

$C.F. = e^{-x}(A\cos 2x + B\sin 2x)$.

$$P.I. = \frac{1}{D^2+2D+5}\cdot xe^x$$

$$= e^x\frac{1}{(D+1)^2+2(D+1)+5}\cdot x$$

$$= e^x\frac{1}{D^2+4D+8}\cdot x$$

$$= e^x\frac{1}{8+4D}\cdot x$$ (as D^2 can be omitted in the denominator for only x occurs in numerator).

$$= \frac{e^x}{8}\left(1 - \frac{D}{2}\right) x$$

$$= \frac{e^x}{8}\left(x - \tfrac{1}{2}\right).$$

$$y = C.F. + P.I.$$

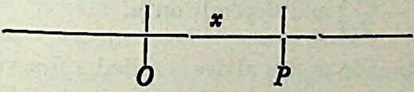
Exercises XCIII.

Solve the following equations :—

1. $(D^2 + 1)\, y = (x^2 + 1)\, e^x$.
2. $(D^2 + 4)\, y = x\, e^{2x}$.
3. $(D^2 - 4D + 3)\, y = e^x \cos 2x$.
4. $(D^2 - 2D + 2)\, y = e^x \cos x$.
5. $(D^2 - 2D + 4)\, y = e^x \sin x$.

§ 74. (*a*) Application to Damped oscillation.

A particle of mass m moves in a straight line under a force mn^2 (distance) towards a fixed point in the straight line and under a small resistance to its motion equal to $m\mu$ (velocity) ; to find the motion.



Let the distance of the moving point P from the fixed point O at time t be x. The equation of motion is

$$m\frac{d^2x}{dt^2} = -mn^2x - m\mu\frac{dx}{dt}$$

i.e., $\ddot{x} + \mu\dot{x} + n^2x = 0.$ (1)

[This is the equation of motion if P moves towards the right so that x is increasing. If P were to move towards the left so that x is decreasing, the resistance is towards the right equal to $m\mu v$, where v is the velocity. But $\frac{dx}{dt}$ is negative, as x decreases when t increases ; $v = -\frac{dx}{dt}$; the resistance $= m\mu\left(-\frac{dx}{dt}\right) \rightarrow$. The equation of motion is thus the same as (1).]

$\therefore$ (1) represents the motion of P whether P moves towards the right or left.

Putting $D = \frac{d}{dt}$, (1) becomes $(D^2 + \mu D + n^2)\, x = 0.$

The auxiliary equation is $m^2 + \mu m + n^2 = 0.$

Case I. $\mu < 2n$; $m = \dfrac{-\mu \pm \sqrt{\mu^2 - 4n^2}}{2}$

$$= \frac{\mu}{2} \pm i\sqrt{n^2 - \frac{\mu^2}{4}}.$$

$$\therefore \quad x = Ae^{-\mu t/2}\left[\cos\sqrt{n^2 - \frac{\mu^2}{4}}\,t + B\right], \tag{2}$$

where A and B are arbitrary constants.

If μ be small, $Ae^{-\mu t/2}$ is a *slowly* varying quantity. Hence (2) represents approximately a simple harmonic motion of period $\dfrac{2\pi}{\sqrt{n^2 - \mu^2/4}}$, whose amplitude $Ae^{-\mu t/2}$ is a slowly decreasing quantity. Such a motion is called a *damped* oscillation and μ measures the damping.

As the period depends on the square of μ, the small frictional resistance has no effect on the period to the first order of approximation. Its effect is seen in the decreasing amplitude $Ae^{-\mu t/2} = A\left(1 - \dfrac{\mu t}{2}\right)$ and depends on μ.

Such a vibration as the above is called a *free* vibration. It is the vibration of a particle which moves under the action of no external periodic force.

By differentiating (2) with respect to t,

$$\dot{x} = -\frac{A\mu}{2}e^{-\mu t/2}\cos\left[\sqrt{n^2 - \frac{\mu^2}{4}}\,t + B\right]$$
$$- Ae^{-\mu t/2}\sqrt{n^2 - \frac{\mu^2}{4}}\sin\left[\sqrt{n^2 - \frac{\mu^2}{4}}\,t + B\right]$$

$= 0$, when

$$\tan\left[\sqrt{n^2 - \frac{\mu^2}{4}}\,t + B\right] = -\frac{\mu}{\sqrt{4n^2 - \mu^2}} = \tan\alpha \text{ (say)}.$$

The solutions of this equation are

$$\sqrt{n^2 - \frac{\mu^2}{4}}\,t_1 + B = \alpha$$
$$\sqrt{n^2 - \frac{\mu^2}{4}}\,t_2 + B = \alpha + \pi$$
$$\sqrt{n^2 - \frac{\mu^2}{4}}\,t_3 + B = \alpha + 2\pi \ldots\ldots$$

$t_1, t_2, t_3 \ldots\ldots$ form an Arithmetical Progression whose common

difference is $\dfrac{\pi}{\sqrt{n^2-\dfrac{\mu}{4}}}$. The corresponding values of the amplitudes are $Ae^{-\mu t_1/2}$, $Ae^{-\mu t_2/2}$, $Ae^{-\mu t_3/2}$ ...... and they form a decreasing G.P. whose common ratio is

$$e^{-\mu/2\,(t_2-t_1)}=e^{\frac{-\mu\pi}{2\sqrt{n^2-\mu^2/4}}}.$$

Case II. $\mu>2n$. The solution of (1) takes a different form

$$x=e^{-\mu t/2}\left[Ae^{\sqrt{\frac{\mu^2}{4}-n^2}\,t}+Be^{-\sqrt{\frac{\mu^2}{4}-n^2}\,t}\right],$$

where A and B are arbitrary constants.

The motion is not oscillatory.

Case III. $\mu=2n$. The auxiliary equation has equal roots.

$$\therefore \quad m=-\frac{\mu}{2} \text{ twice.}$$

$$\therefore \quad x=e^{-\mu t/2}(A+Bt) \text{ by § 70,}$$

where A and B are arbitrary constants.

In this case also, the motion is not oscillatory.

§ 74. (*b*) *A point is moving in a straight line with an acceleration μx towards a fixed point in the straight line and with an additional acceleration L cos pt; to find the motion.*

If the distance of the moving point at time t from the fixed point be x, the equation of motion is

$$\frac{d^2x}{dt^2}=-\mu x+L\cos pt.$$

The $C.F.$ is the solution of $\dfrac{d^2x}{dt^2}=-\mu x$

i.e., $A\cos(\sqrt{\mu}\,t+B)$.

$$P.I.=\frac{1}{D^2+\mu}L\cos pt=\frac{L}{\mu-p^2}\cos pt \text{ if } \mu\neq p^2.$$

$$\therefore \quad x=A\cos(\sqrt{\mu}\,t+B)+\frac{L}{\mu-p^2}\cos pt.$$

If the particle starts from rest at a distance a at zero time, so that $t=0$, $x=a$ and $\ddot{x}=0$, we have

$$a=A\cos B+\frac{L}{\mu-p^2} \qquad \text{(i)}$$

$$\text{and } 0=-A\sqrt{\mu}\sin B \qquad \text{(ii)}$$

$\therefore \quad B = 0$ and $A = a - \dfrac{L}{\mu - p^2}$.

$$\therefore \quad x = \left(a - \frac{L}{\mu - p^2}\right)\cos\sqrt{\mu}\,t + \frac{L}{\mu - p^2}\cos pt.$$

Thus the motion is compounded of two simple harmonic motions whose periods are $\dfrac{2\pi}{\sqrt{\mu}}$ and $\dfrac{2\pi}{p}$. If $\sqrt{\mu}$ be nearly equal to p, the coefficient $\dfrac{L}{\mu - p^2}$ of the disturbing acceleration $L\cos pt$ becomes very great. Thus if the period $\dfrac{2\pi}{p}$ of the disturbing force be nearly equal to that of the free motion $\dfrac{2\pi}{\sqrt{\mu}}$, its effect may be very great even though its absolute magnitude L be comparatively small.

If $p = \sqrt{\mu}$, the $P.I. = \dfrac{L}{D^2 + p^3}\cos pt$

$$= L \text{ Real Part of } \frac{1}{(D + pi)(D - pi)} e^{ipt}$$

$$= L \quad ,, \quad \frac{1}{2pi} \cdot te^{ipt}$$

$$= L \quad ,, \quad -\frac{i}{2p}\,t\,(\cos pt + i\sin pt)$$

$$= \frac{Lt}{2p}\sin pt.$$

$$x = A\cos(pt + B) + \frac{Lt}{2p}\sin pt.$$

If, when $t = 0$, $x = a$ and $\dot{x} = 0$,

$$a = A\cos B \qquad \text{(i)}$$

$$0 = -Ap\sin B \qquad \text{(ii)}$$

$\therefore \quad B = 0$ and $A = a$.

Hence $\quad x = a\cos pt + \dfrac{Lt}{2p}\sin pt.$

As t becomes very large, x and $\dot{x}$ both become large.

§ 75. **Forced vibration.** *A particle, of mass m, is moving in a straight line under force mn^2 (distance) towards a fixed point in the straight line and under a frictional resistance equal to $m\mu$ (velocity) and a periodic force $mL\cos pt$; to find the motion.*

If the distance of the moving point at time t from the fixed point be x, the equation of motion is

$$\ddot{x} + \mu\dot{x} + n^2 x = L \cos pt.$$

To find the $C.F.$, we have to solve $\ddot{x} + \mu\dot{x} + n^2 x = 0$.

The auxiliary equation is $m^2 + \mu m + n^2 = 0$.

$$m = \frac{-\mu \pm \sqrt{\mu^2 - 4n^2}}{2} = \frac{\mu}{2} \pm i\sqrt{n^2 - \frac{\mu^2}{4}} \text{ taking } \mu < 2n.$$

$$\therefore \quad C.F. = Ae^{-\mu t/2} \cos\left[\sqrt{n^2 - \frac{\mu^2}{4}}\, t + B\right].$$

$$P.I. = \frac{1}{D^2 + \mu D + n^2} L \cos pt \text{ where } D = \frac{d}{dt}$$

$$= \frac{L}{n^2 - p^2 + \mu D} \cos pt = L\frac{(n^2 - p^2) - \mu D}{(n^2 - p^2)^2 - \mu^2 D^2} \cos pt$$

$$= L\frac{(n^2 - p^2)\cos pt + \mu p \sin pt}{(n^2 - p^2)^2 + \mu^2 p^2}.$$

Putting $r \cos \epsilon = n^2 + p^2$ and $r \sin \epsilon = \mu p$,

$$r^2 = (n^2 - p^2)^2 + \mu^2 p^2 \text{ and } \tan \epsilon = \frac{\mu p}{n^2 - p^2}.$$

$$P.I. = L\frac{\cos(pt - \epsilon)}{r}.$$

$$\therefore \; x = Ae^{-\mu t/2} \cos\left[\sqrt{n^2 - \frac{\mu^2}{4}}\, t + B\right] + \frac{L}{r}\cos(pt - \epsilon).$$

The motion is compounded of two oscillations; the first (represented by the $C.F.$) is called the *free* vibration and the second ($P.I.$) is called the *forced* vibration.

In particular, if $p = n$, the $P.I.$, *i.e.*, the solution for the forced vibration is $\frac{L}{\mu n} \sin nt$. If μ is, as usual, small, $\frac{L}{\mu n}$ becomes very large. Thus a small periodic force may, if its period be nearly equal to that of the free vibration, produce considerable effects. For example, when soldiers march over a bridge in step, there may arise some danger to the bridge owing to the accumulative effect of their march.

When μ is small, $e^{-\mu t/2}$ gradually decreases as t increases and causes the $C.F.$ to decrease and vanish ultimately. Thus ultimately the free vibration dies out and the forced vibration only prevails.

CHAPTER XVI

EQUATIONS OF THE FIRST ORDER, BUT OF HIGHER DEGREE

§ 76. Type A :—Equations solvable for $\frac{dy}{dx}$.

We shall denote $\frac{dy}{dx}$ hereafter by p.

Let the equation of the first order and of the n^{th} degree in p be $p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots\ldots + P_n = 0$, (1)
where $P_1, P_2, \ldots\ldots, P_n$, denote functions of x and y.

Suppose the first member of (1) can be resolved into factors of the first degree of the form

$$(p - R_1)(p - R_2)(p - R_3) \ldots\ldots (p - R_n).$$

Any relation between x and y which makes any of these factors vanish is a solution of (1). Let the primitives of $p - R_1 = 0$, $p - R_2 = 0$, etc. be

$$\phi_1(x, y, c_1) = 0, \phi_2(x, y, c_2) = 0 \ldots\ldots \phi_n(x, y, c_n) = 0$$

respectively, where $c_1, c_2, \ldots\ldots c_n$ are arbitrary constants. Without any loss of generality, we can replace $c_1, c_2, \ldots\ldots c_n$ by c where c is an arbitrary constant. Hence the solution of (1) is $\phi_1(x\,y\,c)$, $\phi_2(x\,y\,c) \ldots\ldots \phi_n(x\,y\,c) = 0$.

Examples.

Ex. 1. Solve $x^2p^2 + 3xyp + 2y^2 = 0$.

Solving for p, $p = -\frac{y}{x}$ or $= -\frac{2y}{x}$.

$$\frac{dy}{dx} = -\frac{y}{x} \text{ gives } xy = c. \quad (1)$$

$$\frac{dy}{dx} = -\frac{2y}{x} \text{ gives } yx^2 = c. \quad (2)$$

The solution is $(xy - c)(yx^2 - c) = 0$.

Ex. 2. Solve $p^2 + \left(x + y - \frac{2y}{x}\right)p + xy + \frac{y^2}{x^2} - y - \frac{y^2}{x} = 0.$

$$p = \frac{y}{x} - y \quad \text{or} \quad \frac{y}{x} - x.$$

$$\frac{dy}{y} = \left(\frac{1}{x} - 1\right) dx \quad \text{or} \quad \frac{dy}{dx} - \frac{y}{x} = -x.$$

The first equation gives

$$\log\frac{y}{x} = -x + \log c$$

$$y = cx\,e^{-x}. \tag{1}$$

The second equation is linear in y. Hence the solution is

$$ye^{-\int dx/x}$$

$$= ye^{-\log x}$$

$$= \frac{y}{x} = -\int \frac{x}{x}\,dx + c$$

$$= -x + c. \tag{2}$$

The general solution is

$$(y - cx\,e^{-x})\,(y + x^2 - cx) = 0.$$

§ 77. TYPE B :—Let the differential equation (1) of § 76 be put in the form $f(x, y, p) = 0$. When it cannot be resolved into rational linear factors as in § 76, it may be either solved for y or x.

§ 77·1. Equations solvable for y.

$f(x, y, p) = 0$ can be put in the form

$$y = F(x, p). \tag{1}$$

Differentiating with respect to x,

$$p = \phi\left(x, p, \frac{dp}{dx}\right).$$

This, being an equation in the two variables p and x, can be integrated by any of the foregoing methods. Hence we obtain,

$$\psi(x, p, c) = 0. \tag{2}$$

Eliminating p between (1) and (2), the solution is got.

§ 77·2. Equations solvable for x.

Let $f(x, y, p) = 0$ be in this case put in the form

$$x = F(y, p). \tag{1}$$

Differentiating with respect to y,

$$\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right).$$

Integration leads to $\Psi(y, p, c) = 0.$ (2)

Eliminating p between (1) and (2), the solution of (1) is got.

Examples.

Ex. 1. Solve $xp^2 - 2yp + x = 0$.

Solving for y, $y = x\dfrac{(p^2+1)}{2p}$.

Differentiating with respect to x,

$$p = \frac{p^2+1}{2p} + x\,\frac{p^2-1}{2p^2}\,\frac{dp}{dx}$$

$$\frac{p^2-1}{p} = \frac{dp}{dx}x\frac{p^2-1}{p^2}.$$

$$\therefore \frac{dx}{x} = \frac{dp}{p}.$$

Integrating, $p = cx$.

Eliminating p between this and the given equation, the solution is $2cy = c^2x^2 + 1$.

Ex. 2. Solve $x = y^2 + \log p$. (1)

(This is easily solvable for x only.)

Differentiating with respect to y,

$$\frac{1}{p} = 2y + \frac{1}{p}\frac{dp}{dy}.$$

$\dfrac{dp}{dy} + 2py = 1$. This is linear in p and hence

$$pe^{y^2} = \int e^{y^2}\,dy + c. \qquad (2)$$

(It must be noted that the integral on the R.H.S. cannot be integrated in finite terms.)

The eliminant of p between (1) and (2) give the solution.

Note. In the above problem, the solution has not been got explicitly by eliminating p. But we have x and y expressed in terms of a parameter p. This will do.

§ 78. Particular cases of § 77·1.

There are two special cases of equations solvable for y, worthy of note.

§ 78·1. Clairaut's form.

The equation known as Clairaut's is of the form (1)

$$y = px + f(p).$$

Differentiating with respect to x,

$$p = p + \{x + f'(p)\}\frac{dp}{dx}.$$

Either $\dfrac{dp}{dx} = 0$ or $x + f'(p) = 0$.

$$\frac{dp}{dx} = 0 \text{ gives } p = c, \text{ a constant.}$$

$\therefore$ The solution of (1) is $y = cx + f(c)$.

We have to replace p in Clairaut's equation by c. The other factor $x + f'(p) = 0$ taken along with (1) give, on elimination of p, a solution of (1). But this solution is not included in the general solution (2). Such a solution as this is called a *singular* solution.

Examples.

Ex. 1. Solve $y = (x - a)p - p^2$. (B.E. 52)

This is Clairaut's equation ; hence the solution is

$$y = (x - a)c - c^2.$$

Ex. 2. Solve $y = 2px + y^2p^3$.

Putting $X = 2x$ and $Y = y^2$,

$$dX = 2dx \ : \ dY = 2y\,dy.$$

$$\therefore \quad P = \frac{dY}{dX} = yp.$$

The equation transforms into $Y = XP + P^3$.

This is Clairaut's equation ; hence

$$Y = cX + c^3.$$

The solution is $y^2 = 2xc + c^3$.

§ 78·2. We have an extended form of Clairaut's equation of the type $y = xf(p) + \phi(p)$. (1)

Differentiating with respect to x,

$$p = f(p) + \left[xf'(p) + \phi'(p)\right]\frac{dp}{dx}.$$

$$\frac{dx}{dp} + \frac{xf'(p)}{f(p) - p} = \frac{\phi'(p)}{p - f(p)}.$$

This is linear in x and hence gives $F(x, p, c) = 0$.

The eliminant of p between this equation and (1) give the solution of (1).

Example. Solve $y = xp + x(1 + p^2)^{1/2}$.

Differentiating with respect to x,

$$p = p + (1 + p^2)^{1/2} + \frac{dp}{dx}\left[x + \frac{xp}{\sqrt{1 + p^2}}\right]$$

whence $dp\,\dfrac{(\sqrt{1 + p^2} + p)}{(1 + p^2)} + \dfrac{dx}{x} = 0.$

Integrating, $\displaystyle\int \frac{dp}{\sqrt{1+p^2}} + \int \frac{p\,dp}{1+p^2} + \int \frac{dx}{x} = \log c$

i.e., $\log(p+\sqrt{1+p^2}) + \frac{1}{2}\log(1+p^2) + \log x = \log c.$

$\log(p\sqrt{1+p^2}+1+p^2)\,x = \log c.$

$\{p\sqrt{1+p^2}+1+p^2\}\,x = c.$ (2)

Eliminating p between (1) and (2), the solution is got.

§ 79·1. Equations that do not contain x explicitly.

Suppose an equation is of the form

$$f(y, p) = 0. \tag{1}$$

If this is solvable for p, then $p = \phi(y)$ and hence is immediately integrable.

If (1) is solvable for y, so that $y = \phi(p)$, then the method of § 77·1 is applied.

§ 79·2. Equations that do not contain y explicitly.

Let the equation be $f(x, p) = 0.$ (1)

If this is solvable for p, so that $p = \phi(x)$, it is directly integrable.

If (1) is solvable for x, the method of § 77·2 is applied.

§ 79·3. Equations homogeneous in x and y.

Let the equation be $f\left(\frac{y}{x}, p\right) = 0.$ (1)

If this is solvable for p, then $p = F\left(\frac{y}{x}\right)$ and is immediately integrable.

If (1) is solvable for $\frac{y}{x}$, so that $y = xF(p)$, then we proceed as in § 78·2.

Differentiate with respect to x; we have

$$p = F(p) + xF'(p)\frac{dp}{dx}$$

$$\therefore \frac{dx}{x} = \frac{F'(p)\,dp}{p - F(p)}.$$

This is integrable and the eliminant of p between this equation and (1) is the required solution.

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Examples.

Ex. 1. Solve $x^2 = (1 + p^2)$.

$x = \pm\sqrt{1+p^2}$. Here y is explicitly absent.

Differentiate with respect to y.

$$\frac{1}{p} = \frac{p}{\sqrt{1+p^2}}\frac{dp}{dy}$$

$$\therefore \quad dy = \frac{p^2}{\sqrt{1+p^2}}\,dp.$$

$$\text{Hence } y + c = \int \frac{p^2}{\sqrt{1+p^2}}\,dp = \int \frac{p^2+1-1}{\sqrt{1+p^2}}\,dp$$

$$= \int \left(\sqrt{1+p^2} - \frac{1}{\sqrt{1+p^2}}\right) dp$$

$$= \frac{1}{2}\left(p\sqrt{1+p^2} + \sinh^{-1} p\right) - \sinh^{-1} p$$

$$= \frac{1}{2}(p\sqrt{1+p^2} - \sinh^{-1} p). \qquad (2)$$

Eliminating p between (1) and (2), the solution is got.

Ex. 2. Solve $xyp^2 + p\,(3x^2 - 2y^2) - 6xy = 0$. (B.E. 47)

This is homogeneous in x and y and solvable for p.

$$p = \frac{2y}{x} \text{ or } -\frac{3x}{y}.$$

$$\frac{dy}{y} = \frac{2dx}{x} \text{ or } y\,dy + 3x\,dx = 0.$$

$$\therefore \quad y = cx^2 \text{ or } y^2 + 3x^2 = c.$$

The solution is $(y - cx^2)\,(y^2 + 3x^2 - c) = 0$.

Ex. 3. Solve $(xp - y)^2 = a\,(1 + p^2)\,\phi\,(x^2 + y^2)$.
[B.A. (Hons.) 56]

$$(xdy - ydx)^2 = a\,(dx^2 + dy^2)\,\phi\,(x^2 + y^2).$$

Changing to polar coordinate,

$xdy - ydx = r^2 d\theta$ [equating elements of area in cartesian and polar coordinates]

and $dx^2 + dy^2 = ds^2 = dr^2 + r^2 d\theta^2$,
where ds is the element of arc.

$\therefore$ The equation transforms into

$$r^4\,(d\theta)^2 = [dr^2 + r^2\,(d\theta)^2]\,a\phi\,(r^2)$$

$$\therefore \quad d\theta = \frac{dr}{r}\sqrt{\frac{a\phi\,(r^2)}{r^2 - a\phi\,(r^2)}}$$

$$\therefore \quad \theta + C = \int \frac{dr}{r}\sqrt{\frac{a\phi\,(r^2)}{r^2 - a\phi\,(r^2)}}.$$

Exercises XCIV. Solve :

1. $y = px + \dfrac{ap}{(1+p^2)^{1/2}}$. (B.E. 45)
2. $p^2 - 5p + 6 = 0$.
3. $y^2 = (1 + p^2)$.
4. $x(1 + p^2) = 1$.
5. $y = px + \dfrac{a}{p}$.
6. $x^2(y - px) = yp^2$. (B.E. 46)
   (Put $X = x^2$; $Y = y^2$.)
7. $xyp^2 + (y^2 - x^2)p - xy = 0$.
8. $e^{3x}(p - 1) + p^3 e^{2y} = 0$.
9. $y - xp = x + yp$.
10. $y - 2px = x^2 p^4$.
11. $p^2 + 2yp \cot x = y^2$.
12. $x - yp = ap^2$. (B.E. 53)

[*Hint.* Solve for y and differentiate with respect to x; a linear equation in x is got.]

13. $p^2 y + p(x - y) - x = 0$. [B.A. (Hons.) 54]
14. $xp(3y^2 - ax) = y(2y^2 - ax)$. [B.A. (Hons.) 54]

[*Hint.* Put $y = vx$; a linear equation in x is got.]

15. Solve $(px - y)(py + x) = 2p$. (M.Sc. 60)

[*Hint.* Put $x^2 = X$, $y^2 = Y$; the equation reduces to Clairaut's equation.]

§ 80. Linear equations with variable coefficients.

We shall first consider the homogeneous linear equation. A *homogeneous linear equation* of the second order is of the form

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = X \tag{1}$$

where a, b, c are constants and X a function of x.

§ 80·1. *Method* 1. By putting $z = \log x$ or $x = e^z$, this equation can be transformed into one with constant coefficients.

We introduce here an operator called $\theta = x\dfrac{d}{dx}$. Now,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz};\; x\frac{dy}{dx} = \frac{dy}{dz} = Dy \text{ if } D \text{ stands for } \frac{d}{dz}. \tag{i}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) = \frac{D}{x^2}(D - 1)$$

$$\therefore \quad x^2 \frac{d^2y}{dx^2} = D(D-1)\,y. \qquad \text{(ii)}$$

We note that $D = \dfrac{d}{dz} = x\dfrac{d}{dx} = \theta.$

So, putting $x = e^z$ in (1), the equation (1) becomes $\{aD(D-1) + bD + c\}\,y = Z$ (2) where Z is the function of z into which X has been transformed. This equation (2) is a linear equation with constant coefficients and hence the foregoing method can be adopted.

§ 80·2. *Method* 2. Without transforming (1) into a linear equation with constant coefficients, an independent method can be given.

To find the complementary function of (1).

Thus we have to solve

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0.$$

If x^m, for some value of m, be taken as a tentative solution, then, on substitution, we get

$$am(m-1) + bm + c = 0.$$

This, being an equation of the second degree in m, has two roots m_1, m_2. Hence the complementary function of (1) is $C_1 x^{m_1} + C_2 x^{m_2}$, taking the two roots to be distinct.

If, however, a root m_1 be repeated twice putting $m_2 = m_1 + \epsilon$ where $\epsilon \to 0$, the corresponding part of the C.F. is $x^{m_1}(C_1 + C_2 x^{\epsilon}) = x^{m_1}(C_1 + C_2 e^{\epsilon \log x}) = x^{m_1}(C_1 + C_2\,\overline{1 + \epsilon \log x})$, ϵ^2 etc. being neglected as $\epsilon \to 0$. Putting $C_2\epsilon = B$ and $C_1 + C_2 = A$, the part of the C.F., arising from the two equal roots m_1, is $x^{m_1}(A + B\log x)$.

§ 80·3. **To find the Particular Integral.**

The P.I. of $ax^2\dfrac{d^2y}{dx^2} + bx\dfrac{dy}{dx} + cy = X$ (1) is now found. Using $\theta = x\dfrac{d}{dx}$, the first member of (1) can be symbolically written as $f(\theta)\,y$, where $f(\theta) = a\theta(\theta - 1) + b\theta + c$.

$\therefore$ (1) can be written as $f(\theta)\,y = X$.

The P.I. is $\dfrac{1}{f(\theta)}X$, where $\dfrac{1}{f(\theta)}$ is the inverse operator defined as in § 72.

If $f(\theta) \equiv (\theta - \alpha_1)(\theta - \alpha_2)$, the P.I. can be put either as

$$\frac{1}{(\theta - \alpha_1)}\frac{1}{(\theta - \alpha_2)}X$$

$$\text{or}\quad \left(\frac{A_1}{\theta - \alpha_1} + \frac{A_2}{\theta - \alpha_2}\right)X,$$

by the method of partial fractions.

It must be noted that in the first form, the order of the operators is *not* commutative as in § 79·1. Here, the operations indicated by the factors are to be taken in succession, beginning with the first on the right. Thus, the general method of finding the P.I. ultimately depends on the evaluation of $\frac{1}{\theta - \alpha}X$.

§ 80·4. *To find* $\frac{1}{\theta - \alpha}X$.

Let $u = \frac{1}{\theta - \alpha}X$.

By definition of inverse operator, $x\frac{du}{dx} - \alpha u = X$

$$i.e.,\quad \frac{du}{dx} - \frac{\alpha}{x}u = \frac{X}{x}.$$

This equation is linear in u and hence its solution is

$$ux^{-\alpha} = \int x^{-\alpha-1}X\,dx,$$

no constant being added as this is a particular integral.

$$\therefore\quad \mathbf{u = x^{\alpha}\int x^{-\alpha-1}\,X\,dx.}$$

(It is advisable for the student to commit this result to memory.)

§ 80·5. *Special method of evaluating the P.I. when X is of the form x^m.*

$$\theta\, x^m = x\frac{d}{dx}(x^m) = mx^m.$$

$$\theta^2 x^m = x\frac{d}{dx}(mx^m) = m^2x^m.$$

Operating on both sides by $\frac{1}{f(\theta)}$,

$$x^m = f(m)\frac{1}{f(\theta)}x^m.$$

If $f(m) \neq 0$, $\frac{1}{f(\theta)}x^m = \frac{1}{f(m)}x^m.$

If, however, $f(m) = 0$, then $f(\theta) = (\theta - m)\phi(\theta)$ where $\phi(m) \neq 0$. P.I. becomes $\dfrac{1}{\phi(m)} \dfrac{1}{\theta - m} x^m$

$$= \frac{1}{\phi(m)} x^m \int x^{-m-1} x^m \, dx \text{ by the above general method}$$

$$= \frac{1}{\phi(m)} x^m \log x.$$

If m be repeated 2 times in $f(m) = 0$, the P.I. is $\dfrac{x^m (\log x)}{2\,!}$ where $f(m) = (\theta - m)^2$.

Examples.

Ex. 1. Solve $3x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = x.$

Putting $z = \log x$ and $D = \dfrac{d}{dz}$, the equation becomes

$$[3D(D-1) + D + 1]\,y = e^z.$$

The auxiliary equation is $3m^2 - 2m + 1 = 0$.

$$m = 1 \pm \sqrt{2}\,i.$$

$$\text{C.F.} = e^z (A \cos \sqrt{2}\, z + B \sin \sqrt{2}\, z)$$
$$= x\,\{A \cos(\sqrt{2} \log x) + B \sin(\sqrt{2} \log x)\}$$

$$\text{P.I.} = \frac{1}{3D^2 - 2D + 1} e^z$$
$$= \frac{1}{3 - 2 + 1} e^z \text{ by § 73 . } a$$
$$= \frac{e^z}{2} = \frac{x}{2}.$$

$$y = \text{C.F.} + \text{P.I.}$$
$$= x\,[A \cos(\sqrt{2} \log x) + B \sin(\sqrt{2} \log x) + \tfrac{1}{2}].$$

Ex. 2. Solve $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = \log x.$

Putting $z = \log x$ and $D = \dfrac{d}{dz}$, the equation becomes

$$[D(D-1) + D + 1]\,y = z$$
$$i.e., \quad (D^2 + 1)\,y = z.$$

The auxiliary equation is $m^2 + 1 = 0$.

$$\text{C.F.} = A \cos z + B \sin z$$
$$= A \cos \log x + B \sin \log x.$$

$$\text{P.I.} = \frac{1}{D^2+1} z = (1+D^2)^{-1} z$$
$$= (1 - D^2 + \ldots) z = z = \log x.$$
$$\therefore \quad y = A \cos \log x + B \sin \log x + \log x.$$

Ex. 3. Solve $x^2 \dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = \dfrac{1}{(1-x)^2}$. (B.E. 53)

Putting $z = \log x$ and $D = \dfrac{d}{dz}$ the equation becomes

$$[D(D-1) + 3D + 1] y = \frac{1}{(1-x)^2}$$

i.e., $(D^2 + 2D + 1) y = \dfrac{1}{(1-x)^2}$.

The auxiliary equation is $(m+1)^2 = 0$.

$$\therefore \quad m = -1 \text{ twice.}$$

$$\text{C.F.} = e^{-z}(A + Bz) = \frac{1}{x}(A + B \log x).$$

$$\text{P.I.} = \frac{1}{(\theta+1)^2} \frac{1}{(1-x)^2} \text{ changing } D \text{ to the operator } \theta = x\frac{d}{dx}$$

$$= \frac{1}{\theta+1} x^{-1} \int \frac{dx}{(1-x)^2} \text{ by § 80·4}$$
$$= \frac{1}{\theta+1} \frac{1}{x} \frac{1}{1-x}$$
$$= x^{-1} \int \frac{dx}{x(1-x)} = x^{-1} \int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx$$
$$= \frac{1}{x} \log \frac{x}{1-x}.$$
$$y = \text{C.F.} + \text{P.I.}$$

Ex. 4. Solve $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = e^x$. (B.E. 44)

Putting $z = \log x$ and $D = \dfrac{d}{dz}$, the equation becomes

$$(D^2 + 3D + 2) y = e^x.$$

The auxiliary equation is $m^2 + 3m + 2 = 0$.

$$\therefore \quad m = -1 \text{ or } -2.$$

$$\text{C.F.} = Ae^{-z} + Be^{-2z} = A\,x^{-1} + B\,x^{-2}.$$

$$\text{P.I.} = \frac{1}{(\theta+1)(\theta+2)}\,e^x \text{ where } D = \theta = x\frac{d}{dx}$$

$$= \left[\frac{1}{\theta+1} - \frac{1}{\theta+2}\right]e^x$$

$$= x^{-1}\int e^x\,dx - x^{-2}\int xe^x\,dx \text{ by § 80·4}$$

$$= x^{-1}e^x - x^{-2}(x\,e^x - e^x)$$

$$= x^{-2}e^x.$$

$$y = A\,x^{-1} + B\,x^{-2} + x^{-2}e^x.$$

Ex. 5. Solve $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = \dfrac{\log x\sin(\log x) + 1}{x}$.

Putting $z = \log x$ and $D = \dfrac{d}{dz}$, the equation becomes

$$(D-1)^2 y = \frac{(z\sin z + 1)}{e^z}.$$

The auxiliary equation is $(m-1)^2 = 0$; $m = 1$ twice.

$$\text{C.F.} = e^z(A + Bz) = x(A + B\log x).$$

$$\text{P.I.} = \frac{1}{(D-1)^2}[z\sin z + 1]\,e^{-z}$$

$$= e^{-z}\frac{1}{(D-2)^2}[z\sin z + 1] \text{ by § 73 } (d)$$

$$= x^{-1}\left[\text{Imaginary Part of } \frac{1}{(D-2)^2}ze^{iz} + \frac{1}{4}\right]$$

$$= x^{-1}\left[\text{I.P. of } e^{iz}\frac{1}{(D+i-2)^2}z + \frac{1}{4}\right]$$

$$= x^{-1}\left[\text{I.P. of } e^{iz}\frac{1}{(i-2)^2}\left\{1 - \frac{2D}{i-2}\right\}z + \frac{1}{4}\right]$$

$$= x^{-1}\left[\text{I.P. of } e^{iz}\frac{1}{3-4i}\left\{z + \frac{2}{5}(i+2)\right\} + \frac{1}{4}\right]$$

$$= x^{-1}\left[\frac{1}{25}\left\{\left(3z + \frac{4}{5}\right)\sin z + \left(4z + \frac{6}{5}\right)\cos z\right\} + \frac{1}{4}\right]$$

$$= \frac{1}{100x} + \frac{6}{125x}\cos(\log x) + \frac{4}{125x}\sin\log x$$

$$+ \frac{1}{25x}\log x\left\{4\cos(\log x) + 3\sin(\log x)\right\}.$$

$$y = \text{C.F.} + \text{P.I.}$$

Exercises XCV. Solve :

1. $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - 3y = x^2.$ (B.Sc. M. 47)

2. $x^2 \frac{d^2y}{dx^2} + 8x \frac{dy}{dx} + 12y = x^4.$ [B.A. (Hons.) 41]

3. $x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 6x^2 + 2x + 1.$ (B.E. 45)

4. $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + 2y = x^2.$ (B.Sc. M. 48)

5. $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = x^2.$ [B.A. (Hons.) 52]

6. $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} - 5y = \sin \log x.$ (B.E. 48)

7. $x^2 \frac{d^2y}{dx^2} + 7x \frac{dy}{dx} + 13y = \log x.$ [B.A. (Hons.) 50]

8. $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = \log x.$

9. $x^2 \frac{d^5y}{dx^5} - 4x \frac{d^4y}{dx^4} + 6 \frac{d^3y}{dx^3} = 4.$ [B.A. (Hons.) 38]

[*Hint.* Put $v = \frac{d^3y}{dx^3}$ and proceed.]

10. $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = \log x + x.$

§ 81. Equations reducible to the linear homogeneous equation.

Consider an equation of the form

$$(a + bx)^2 \frac{d^2y}{dx^2} + (a + bx)\, p_1 \frac{dy}{dx} + p_2 y = X$$

where p_1, p_2 are constants and X is any function of x.

Putting $z = a + bx$, the equation transforms into

$$z^2 \frac{d^2y}{dz^2} + \frac{p_1}{b} z \frac{dy}{dz} + \frac{p^2}{b^2} y = \frac{1}{b^2} X\left(\frac{z-a}{b}\right).$$

This is a linear homogeneous equation and can be solved by the methods outlined in § 80.

Example. Solve

$$(5 + 2x)^2 \frac{d^2y}{dx^2} - 6\,(5 + 2x) \frac{dy}{dx} + 8y = 6x. \quad \text{(B.E. 51)}$$

Putting $z = 5 + 2x$, the equation becomes

$$4z^2 \frac{d^2y}{dx^2} - 12z \frac{dy}{dx} + 8y = 3(z-5).$$

Putting $u = \log z$ and $D = \frac{d}{du}$, the equation is now transformed into $(4D^2 - 16D + 8)\,y = 3(e^u - 5)$.

The auxiliary equation is $m^2 - 4m + 2 = 0$.

$\therefore \quad m = 2 \pm \sqrt{2}$.

$$\therefore \quad \text{C.F.} = e^{2u}\left(Ae^{\sqrt{2}u} + Be^{-\sqrt{2}u}\right)$$
$$= z^2\left(Az^{\sqrt{2}} + Bz^{-\sqrt{2}}\right)$$
$$= (5+2x)^2\left\{A(5+2x)^{\sqrt{2}} + B(5-2x)^{-\sqrt{2}}\right\}$$

$$\text{P.I.} = \frac{3}{4(D^2 - 4D + 2)}(e^u - 5)$$
$$= -\frac{3}{4}e^u - \frac{15}{8}$$
$$= -\frac{3}{4}z - \frac{15}{8} = -\frac{3}{2}x = \frac{45}{8}$$

$$y = \text{C.F.} + \text{P.I.}$$

Exercises XCVI. Solve :

1. $(x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dv}{dx} + 6y = x.$ (B.E. 50)

2. $(x+1)^2 \frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2 + x + 1.$

3. $(x+a)^2 \frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = x.$

4. $(1+2x)^2 \frac{d^2y}{dx^2} + (1+2x)\frac{dy}{dx} + y = 8(1+2x)^2.$

(Nagpur M.A. 50)

ANSWERS TO EXERCISES

CHAPTER I

Exercises I (Pages 5-6)

1. $f(0) = 3$; $f(1) = -2$; $f(2) = -3$; $f(\frac{1}{2}) = 0$; $f(3) = 0$.
2. $a^6 + 2a^2 + 1$. 4. $2(1 - x)$.
7. $f(y, x) = ay^2 + 2hxy + bx^2$; $f(x, x) = x^2(a + 2h + b)$; $f(y, y) = y^2(a + 2h + b)$.
9. Odd functions (1), 3 (*a*), (*c*), (*d*), (*f*), 4 (*b*).
Even functions (2), 3 (*b*), 3 (*e*), 4 (*a*).

Exercises II (Pages 16-17)

1. $\frac{d}{b}$. 2. $\frac{1}{5}$. 3. $\frac{2}{5}$. 4. 4. 5. 12. 6. $-\frac{1}{4}$. 7. $+\frac{5}{4}$.
8. 2. 9. 3. 10. $\frac{4}{3}$. 11. 1. 12. 6. 13. $\frac{1}{2}$. 14. $-\frac{1}{(2a)^{1/2}}$.
15. 1. 16. -8. 17. $\frac{1}{2}$. 18. $\frac{1}{2}$. 19. $\frac{1}{3}$. 20. $\frac{a}{b}$. 21. $\frac{a}{b}$.
22. $\frac{m}{n}$. 23. $\frac{m^2}{n^2}$. 24. 1. 25. 1. 26. $\frac{1}{2}$. 27. $\frac{1}{2}$. 28. 0. 29. $\frac{1}{2}$.

CHAPTER II

Exercises III (Pages 24-25)

1. $2x - 3$. 2. $8x - 9$. 3. $3lx^2 + 2mx + n$. 4. $12x^3 + 12x$.
5. $54x^8 - 2 - \frac{1}{x^2}$. 6. $2na\,x^{2n-1} + nb\,x^{n-1}$. 7. $-\frac{9}{x^4} + \frac{4}{x^3} - \frac{1}{x^2}$.
8. $1 - \frac{6}{x^3}$. 9. $\frac{2}{3}x^{-1/3}$. 10. $\frac{2}{3}x^{1/2} + \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$.
11. $-\frac{10}{3x^{5/3}} - \frac{3}{4x^{5/4}} - \frac{1}{2x^{1/2}}$. 12. $6x^2 + 2x - 6$.
13. $\frac{162}{x^7} - \frac{270}{x^6} + \frac{144}{x^5} - \frac{24}{x^4}$. 14. $-3x^2 + 3 + \frac{3}{x^2} - \frac{3}{x^4}$.
15. $\frac{3}{2}x^{1/2} - 4x + \frac{1}{2x^{3/2}}$. 16. $e^x + \cos x$. 17. $5\cos x + \frac{1}{x}$.
18. $\frac{3}{x} - e^x + 7\sin x$. 19. $5e^x - \frac{1}{x} + \frac{2}{3}x^{-1/3}$. 20. $\frac{1}{2}e^x - mx^{m-1}$.

21. $\frac{1}{x} - 1.$ 22. $\frac{8}{x} + \cos x.$ 23. $\sqrt{2} \cos x + 20x^4 + \frac{12}{x^5}.$

24. $n\, a^n\, x^{n-1} + \frac{bn}{x}.$ 25. $8x + 3 \sin x + e^x + 2 \cos x.$

Exercises IV (Pages 29-30)

1. $6x - 4.$ 2. $-8x^3 - 2x.$ 3. $x^2 \cos x + 2x \sin x.$

4. $63x^2 - 18x + 28.$ 5. $\frac{\cos x}{x} - \sin x \log x.$

6. $(2 - x^2) \cos x - 4x \sin x.$ 7. $\frac{4(2 + \log x)}{\sqrt{x}}.$

8. $96x^3 + 132x^2 - 92x - 33.$ 9. $e^x (\sin x + x \sin x + x \cos x).$

10. $4x^3 + 9x^2 + 18x + 27.$ 11. $\frac{x \cos x - \sin x}{x^2}.$

12. $-\frac{x \sin x + 2 \cos x}{x^3}.$ 13. $\frac{4x}{(1 - x^2)^2}.$

14. $\frac{3 \operatorname{cosec} x\, (2 \operatorname{cosec} x - 7 \cot x + 3)}{(7 + 3 \cot x)^2}.$ 15. $\frac{\cos x\, (1 - \tan^3 x)}{(1 + \tan x)^2}.$

16. $\frac{8x \sec^2 x - 18 \sec^2 x - 12 - 8 \tan x}{(4x - 9)^2}.$

17. $\frac{-2(x^2 + 6)}{(x^2 - x - 6)^2}.$ 18. $\frac{6(x^2 - 2)}{(x + 1)^2 (x + 2)^2}.$

19. $\frac{6x^9 - 4x^8 + 5x^7 + 12x^5 - 6x^3 - 8x^2 + 3x - 2}{x^3 (2x^3 - x + 1)^2}.$

20. $$\frac{e^x \{ x^4 (5 - x) + \sec x\, (1 - \tan x) \}}{(\sec x - x^5)^2} + \frac{1 - x^5 \cos x + 5x^4 \sin x - \tan^2 x}{(\sec x - x^5)^2}.$$

21. $$\frac{e^x \{ 1 + \sec x\, (1 - \tan x) \}}{(1 + \sec x)^2} + \frac{\left(\frac{(1 + \sec x)}{x} - \log x \,.\, \sec x \tan x \right)}{(1 + \sec x)^2}.$$

Exercises V (Pages 32-33)

1. $3 \cos 3x.$
2. $2 \tan x \sec^2 x.$
3. $-\frac{1}{3} \operatorname{cosec} \frac{x}{3} \cot \frac{x}{3}.$
4. $2 \sec^2 x \tan x.$
5. $-3 \operatorname{cosec}^3 x \cot x.$
6. $2e^{2x}.$
7. $\frac{2}{2x + 3}.$
8. $-4 \sin (4x - 3).$

9. $-n\cot^{n-1}x\operatorname{cosec}^2 x$.

10. $\dfrac{1}{2}\dfrac{\sec^2 x}{\sqrt{\tan x}}$.

11. $an\,x^{n-1}\cos(b+ax^n)$.

12. $\dfrac{2ax+b}{2(ax^2+bx+c)^{1/2}}$.

13. $\dfrac{2}{3}b(a+bx)^{-1/3}$.

14. $-\dfrac{\sin(\log x)}{x}$.

15. $\dfrac{1}{2}(1+\sin x)^{-1/2}\cos x$.

16. $e^x\sec^2(e^x)$.

17. $-\sin x\cos(\cos x)$.

18. $-\tan x$.

19. $2n(ax^2+2bx+c)^{n-1}(ax+b)$.

20. $(x+a)^{m-1}(x+b)^{n-1}\{x(m+n)+mb+na\}$.

21. $2(x^2-1)^3(2x+1)^2(11x^2+4x-3)$.

22. $\dfrac{-(3x+1)}{2\sqrt{1+x}}$.

23. $(a-x)^{n-1}(a-x-nx)$.

24. $\dfrac{x-1}{(2x-1)^{3/2}}$.

25. $\dfrac{n\,x^{n-1}(1-x^2)}{(x^2+1)^{n+1}}$.

26. $-\dfrac{1}{(1-x)^{1/2}(1+x)^{3/2}}$.

27. $-\dfrac{2x}{(x^2+1)^{1/2}(x^2-1)^{3/2}}$.

28. $3\sin 6x$.

29. $\dfrac{(15x^2+5x-12)(3x+5)^3}{(x^2-1)^{1/2}}$.

30. $\sec 5x(5\cos^2 3x\tan 5x-3\sin 6x)$.

31. $\dfrac{n(x-2a-b)(a-x)^{(n-2)/2}}{2(b+x)^{2n+1}}$.

32. $2\tan x\sec^2 x$.

33. $\sin^{m-1}x\cos^{n-1}x(m\cos^2 x-n\sin^2 x)$.

34. $m\cos nx\cos mx-n\sin nx\sin mx$.

35. $-(\cot^3 7x\sin x+21\cos x\cot^2 7x\operatorname{cosec}^2 7x)$.

36. $2\sec^2 4x\operatorname{cosec} 2x(4\tan 4x-\cot 2x)$.

37. $2\sec x$.

38. $\dfrac{1-\sin x+\cos x}{(1+\cos x)(1-\sin x)}$.

39. $2e^{2x-3}(\sin 2x+\cos 2x)$.

40. $\dfrac{3e^{3x}}{b+e^{3x}}+\dfrac{e^x}{a-e^x}$.

41. $-e^{-x}[3\sin x+\cos x]$.

42. $e^{2x}\left[4\log(2x+3)\cos 4x+2\log(2x+3)\sin 4x+\dfrac{2\sin 4x}{2x+3}\right]$.

43. $\dfrac{e^x}{x}\cos(e^x\log x)\{(1+x\log x)\}$.

44. $-\sin(\log x\cot x)\left(-\operatorname{cosec}^2 x\log x+\dfrac{\cot x}{x}\right)$.

45. $4 \cot 4x - 3$.

46. $-e^{4x} \operatorname{cosec} x (4 - \cot x) \operatorname{cosec}^2 (e^{4x} \operatorname{cosec} x)$.

47. $\dfrac{1}{x \log x}$.

48. $e^{2x} [2(x+1) x \log(\operatorname{cosec} x) - x^2 \cot x]$.

49. $e^{2x} [\frac{7}{2} \cos 7x + \sin 7x - \frac{1}{2} \cot x - \sin x]$.

50. $x^2 e^{3x} [8x \cos 8x + 3x \sin 8x + 3 \sin 8x]$.

Exercises VI (Pages 38-39)

1. $\dfrac{1}{2\sqrt{x}(1+x)}$.

2. $\dfrac{-1}{\sqrt{(x-1)(2-x)}}$.

3. $\dfrac{3}{\sqrt{-9x^2+6x+15}}$.

4. $\dfrac{2x}{x^4+1}$.

5. $\dfrac{2 \sin^{-1} x}{\sqrt{1-x^2}}$.

6. $\dfrac{x}{1+x^2} + \tan^{-1} x$.

7. $\sec^{-1} x + \dfrac{1}{\sqrt{x^2-1}}$.

8. $\dfrac{e^x}{\sin^{-1} e^x \sqrt{1-e^{2x}}}$.

9. $-1 - \dfrac{x \cos^{-1} x}{\sqrt{1-x^2}}$.

10. $\dfrac{e^x}{3(1+e^{2x})\{\tan^{-1}(e^x)\}^{2/3}}$.

11. -1.

12. $\dfrac{xe^{ax}}{1+e^{2ax}}$.

13. $\dfrac{1}{1+x^2}$.

14. $\dfrac{-3x^2}{1+x^6}$.

15. $\dfrac{x+1}{x^2\sqrt{x^2-1}}$.

16. $\dfrac{1}{x\{1+(\log x)^2\}}$.

17. $\dfrac{1}{x \log x \sqrt{(\log x)^2 - 1}}$.

18. $\dfrac{-2}{\sqrt{e^{4x+2}-1}}$.

19. $\dfrac{\sqrt{a^2-b^2}}{a+b \sin x}$.

20. $-\dfrac{\sqrt{b^2-a^2}}{b+a \cos x}$.

21. $\dfrac{\sqrt{a^2-b^2}}{a+b \cos x}$.

22. $\dfrac{3 \operatorname{cosec} 3x \cot 3x}{1+\operatorname{cosec}^2 3x}$.

23. $2 \cosh 2x$.

24. $-\frac{1}{3} \operatorname{cosech} \frac{x}{3} \coth \frac{x}{3}$.

25. $3 \coth 3x$.

26. $\dfrac{5}{\sinh 5x \cosh 5x}$.

27. $\dfrac{\frac{1}{2} \operatorname{sech}^2 \frac{x}{2}}{1+\tanh^2 \frac{x}{2}}$.

28. $\dfrac{3}{\sqrt{16+9x^2}}$.

29. $\dfrac{4x}{\sqrt{(2x^2-3)^2-25}}$.

30. $\dfrac{1}{2 \cos x}$.

31. $\dfrac{-\sqrt{2}}{(1+x)\sqrt{1+x^2}}$.

32. $\dfrac{1}{\sqrt{2}\sqrt{1+x}\,(1-x)}$. 33. $\log 2$.

36. $(\tan^{-1} x)\left\{\cos x \tan^{-1} x + \dfrac{2 \sin x}{1+x^2}\right\}$.

37. $\dfrac{8+3x^2+10x^3}{2\sqrt{(2+x^3)\{1-(2x+1)^2(2-x^3)\}}}$. 38. $\dfrac{3}{1+x^2}$.

39. $\dfrac{\log x}{x(2+\log^2 x)\sqrt{1+\log^2 x}}$.

40. $\dfrac{8-3x^2-10x^3}{2\sqrt{(2-x^3)\{1-(2x+1)^2(2-x^3)\}}}$.

Exercises VII (Page 41)

1. $\dfrac{2x}{\sqrt{1+x^3}}(1-x^2)^{-3/2}$.

2. $\sin x \sin 2x \sin 3x \sin 4x\,(\cot x + 2\cot 2x$
$+ 3\cot 3x + 4\cot 4x)$.

3. $\dfrac{a^2+2x^2}{(a^2-x^2)^{5/2}}$. 4. $e^x\{(x+1)\sin x + x\cos x\}$.

5. $x^x(1+\log x)$.

6. $\dfrac{x^4\sqrt[3]{x^2+4}}{\sqrt{4x^2-7}}\left\{\dfrac{4}{x}+\dfrac{2x}{3(x^2+4)}-\dfrac{4x}{4x^2-7}\right\}$.

7. $\dfrac{2x}{\sqrt{(1+x^2)(1-x^2)^{3/2}}}\left\{\dfrac{1}{x}-\dfrac{x}{1+x^2}+\dfrac{3x}{1-x^2}\right\}$.

8. $\dfrac{e^{x^2}\tan^{-1} x}{\sqrt{1+x^2}}\left[2x+\dfrac{1}{(1+x^2)\tan^{-1} x}-\dfrac{x}{1+x^2}\right]$.

9. $x^{5+\sin x}\left[\dfrac{5+\sin x}{x}+\cos x \log x\right]$.

10. $\sin x \log x\, e^x (a^2-x^2)^{3x+7}\left\{\cot x + \dfrac{1}{x\log x}\right.$
$\left.+ 3\log(a^2-x^2) + 1 - \dfrac{2x(3x+7)}{a^2-x^2}\right\}$.

11. $\dfrac{x^3\sqrt{2+3x}}{(2+x)(1-x)}\left[\dfrac{3}{x}+\dfrac{3}{2(2+3x)}+\dfrac{x}{1-x}-\dfrac{1}{2+x}\right]$.

12. $(\log x)^x\left[\dfrac{1}{\log x}+\log(\log x)\right]$.

13. $e^{-x}\sin^n x \cos^m x\,[n\cot x - m\tan x - 1]$.

13. $\dfrac{\cos x}{2\sqrt{1+\sin x}\,(1+\sqrt{1+\sin x})}$. 14. $-\dfrac{1}{2\sqrt{x(1-x)}}$.

15. $6\sec^2(3x+4)\tan(3x+4)$. 16. $\dfrac{1}{\sqrt{1-x^2}}e^{\tan^{-1}\frac{x}{\sqrt{1-x^2}}}$.

17. $\cot x$. 18. $\dfrac{2(1-x^2)}{1+x^2+x^4}$. 19. $\dfrac{1}{1+x^2}$.

20. $4\operatorname{cosec}4x\log_a e$. 22. $(\sin x)^{\log x}\left[\cot x\log x+\dfrac{\log\sin x}{x}\right]$.

23. $\dfrac{1+3x^2-2x^4}{(1-x^2)^{3/2}}$. 24. $\dfrac{1}{4\sqrt{e^{2x}-1}}$.

25. $\dfrac{-(2x^2-16x+25)}{(2x-1)^2(x-3)^2}$. 26. $\tan^{-1}x+\dfrac{x}{1+x^2}$.

27. $\dfrac{(a^2-b^2)\sin x}{(a+b\cos x)(b+a\cos x)}$. 28. $\dfrac{4x^2+1}{x^2(1+x^2)^{3/2}}$.

29. $\dfrac{5^x(x+2)^3}{(x+3)^2}\left[\log 5+\dfrac{3}{x+2}-\dfrac{2}{x+3}\right]$.

30. $\dfrac{x^3e^{bx}\sec x}{x^2+1}\left[\dfrac{3}{x}+b+\tan x-\dfrac{2x}{x^2+1}\right]$.

31. $e^x x^{ex}\left\{\log x+\dfrac{1}{x}\right\}$. 32. $x^x(1+\log x)+\dfrac{1}{x}\cos(\log x)$.

33. $\dfrac{3(x^2-1)^2}{x^6+15(x^4+x^2)+1}$. 34. $\dfrac{a-2x}{(a+4x)\sqrt{2ax-x^2}}$.

35. $\dfrac{(1-x)^{4/3}}{(3x+2)^{1/2}}\left[-\dfrac{4}{3(1-x)}-\dfrac{3}{2(3x+2)}\right]$.

36. $\dfrac{-4x}{4x^4+1}\left\{\tan^{-1}\left(\dfrac{2x^2+a}{(2ax^2-1)}\right)\right\}^{-1}$.

37. $\left(\dfrac{a}{x}\right)^x\left\{\log\left(\dfrac{a}{x}\right)-1\right\}$.

38. $\left[\tan^{-1}(x^2+1)\right]^{\sin x}\left\{\cos x\log\tan^{-1}(x^2+1)+\dfrac{2x\sin x}{\tan^{-1}(x^2+1)}\cdot\dfrac{1}{1+(x^2+1)^2}\right\}$.

39. $-\dfrac{2(5acx+3ad+2bc)}{3(ax+b)^3(cx+d)^{7/3}}$. 40. $\dfrac{(x+1)(x-1)}{(x+2)(x-2)}$.

41. $\dfrac{(x+2)^2\sin x}{\sqrt{2x-1}}\left[\dfrac{2}{x+2}+\cot x-\dfrac{1}{2x-1}\right]$.

42. $1 + \log(\log x) + \log x \log(\log x)$. 43. $\dfrac{1}{(1-x^2)^{3/2}}$.

44. $\dfrac{1}{2(2+x)\sqrt{1+x}}$.

45. $\dfrac{-(x^2+6x+14)}{(x-2)\sqrt{(x-2)^6-(x^2+x+4)^2}}$.

46. $\dfrac{4x}{\sin\{2(x^2+a^2)\}}$. 47. $\log_{10}(ex)$.

48. $\dfrac{-\sin x}{2\cos^2 x + 2\cos x + 1}$. 49. $x^{(\log x)^2}\left[\dfrac{3(\log x)^2}{x}\right]$.

50. $x\cos^{-1} x\left[\dfrac{\cos^{-1} x}{x} - \dfrac{\log x}{\sqrt{1-x^2}}\right]$.

51. $\dfrac{(\sin x)^{\cos x}}{\sin x}[\cos^2 x - \sin^2 x \log \sin x]$.

52. $x^3(x^2+4)^x\left\{\dfrac{3}{x} + \dfrac{2x^2}{x^2+4} + \log(x^2+4)\right\}$.

53. $\dfrac{-1+\cos x-\sin x}{2\sqrt{1+\sin x}\,(1-\cos x)^{3/2}}$. 54. $\dfrac{\sqrt{a^2-b^2}}{2(a+b\cos x)}$.

55. $(b^2-x^2)^{3x}\left[3\log(b^2-x^2) - \dfrac{6x^2}{b^2-x^2}\right]$.

56. $\dfrac{2}{(2x^2-1)\sqrt{1-x^2}}$.

57. $x(1+\log x)^{x^2}\left[\dfrac{1}{1+\log x} + 2\log(1+\log x)\right]$.

58. $-\dfrac{2}{x^2+1}$.

59. $\dfrac{e^{x^2}}{\sqrt{(x-1)(2-x)}}\left[2x - \dfrac{1}{2(x-1)} + \dfrac{1}{2(2-x)}\right]$.

60. $\dfrac{(5x^2+6x-12)}{2\sqrt{x+3}}$. 61. $\dfrac{2\cos 2x - \sin 2x}{e^x}$.

62. $\dfrac{1}{\sqrt{1-x^2}}$. 63. $\dfrac{-x}{\sqrt{1-x^4}}$.

64. $\dfrac{1}{\sqrt{1-x^2}}$. 65. $\dfrac{1}{\sqrt{x^2+a^2}}$.

66. $(a^2+x^2)^x\left[\dfrac{2x^2+(a^2+x^2)\log(a^2+x^2)}{a^2+x^2}\right]$.

67. $2\sqrt{a^2+x^2}$. 68. $-\dfrac{\sqrt{a^2-x^2}}{x}$.

69. $\dfrac{2x+5}{x^2+5x}\log_{10}e$. 70. $\dfrac{\cos a}{x^2+2x\sin a+1}$.

71. $\dfrac{e^{\sin x}}{\sin^2(x^n)}\left\{\sin(x^n)\cos x-nx^{n-1}\cos(x^n)\right\}$.

72. $2x\,4^x\tan(x^2)+4^x\log 4\log(\sec x^2)$.

73. $10^{\log\sin x}\cot x\log_e 10$.

74. $(ax^2+bx+c)^{\cos x}\left[\dfrac{(2ax+b)\cos x}{ax^2+bx+c}-\sin x\log(ax^2+bx+c)\right]$.

75. $-\dfrac{1}{2}$. 76 $\dfrac{2}{x}$. 77. $-\dfrac{1}{2}$.

78. $-\dfrac{5}{4}\dfrac{a}{x^{9/4}}+\dfrac{b}{4}\dfrac{1}{x^{3/4}}$. $a:b=\sqrt{5}:1$.

79. $\dfrac{-x(x^3+3x+2)}{2(2+x^2-x^3)\sqrt{(1-x^3)(1+x^2)}}$.

82. $\dfrac{-2x}{1+x^2}$. 83. $x^y\left[\dfrac{y}{x}+\log x\,.\left(\dfrac{dy}{dx}\right)\right]$.

84. $x\sqrt{\dfrac{1+x^2}{1-x^2}}\left[\dfrac{1}{x}+\dfrac{2x}{1-x^4}\right]$.

85. $\dfrac{x^6-13x^4+29x^2-27}{x^2(x+3)^2}$. 86. $\dfrac{1}{\log 2}$. 88. $\dfrac{x^2}{1-x^4}$.

94. (i) $\left(1+\log\dfrac{x}{n}\right)\left(\dfrac{x}{n}\right)^{nx}\left\{\dfrac{1}{x\left(1+\log\dfrac{x}{n}\right)}+n\left(1+\log\dfrac{x}{n}\right)\right\}$.

(ii) $\dfrac{\sqrt{b^2-a^2}}{a+b\cos x}$.

(iii) $x^{\log\tan x}\left\{\dfrac{\log x}{\sin x\cos x}+\dfrac{\log\tan x}{x}\right\}$.

95. (i) $x^{\sin x}\left[\cos x\log x\dfrac{\sin x}{x}\right]+(\sin x)^x\,[\log\sin x+x\cot x]$.

(ii) $-\dfrac{x(x^2+y^2-3ax)}{y(x^2+y^2-3ay)}$.

(iii) $\dfrac{x-(x+y)\,y\cot x}{(x+y)\log\sin x-1}$.

96. $\dfrac{2x^4+y^3}{x^3}$.

97. $\dfrac{bx^2y}{y^2(x^2+y^2)\sec^2\dfrac{y}{b}+bx^3}$.

98. $\dfrac{y^2(1+\log x)}{1-\log y}$.

99. $\dfrac{\log\sin y+y\tan x}{\log\cos x-x\cot y}$.

100. $\dfrac{\{bx(x-y)-ay\}y}{x(a+bx)(x-y)}$.

101. $\dfrac{2(x+y)\cos(x+y)^2}{1-2(x+y)\cos(x+y)^2}$.

102. $\dfrac{1}{\sqrt{(1-x)^2\{1-(\sin^{-1}x)^2\}}}$.

103. $\dfrac{1+5\sqrt{x}+3x-x\sqrt{x}-2x^2}{2\sqrt{x}(1+x)(1+x^2)}$.

104. $x^{x-1}(1+\log x)x^{x-1}\{1+(1+\log x)^2\log(1+\log x)\}$.

105. $[\log(\sec x+\tan x)]^{\cot x}\operatorname{cosec} x$

$[1-\operatorname{cosec} x\log(\sec x+\tan x)]$.

106. $\dfrac{dy}{dx}=y\left[\dfrac{1}{2}\cot\dfrac{x}{2}+\dfrac{x}{2}\cot\dfrac{x^2}{2^2}+\ldots+\dfrac{nx^{n-1}}{2^n}\cot\dfrac{x^n}{2^n}\right]$

where $y=$ the given function.

107. $\dfrac{1}{2}\,\dfrac{e^{x/2}\sqrt{e^x+e^{-x}}+e^x-e^{-x}}{e^{x/2}+\sqrt{e^x+e^{-x}}}$.

108. $-nx^{n-1}e^{x^n}\tan\left(e^{x^n}\right)$.

109. $\dfrac{1}{2}\left\{\dfrac{1}{\sqrt{x^2-a^2}}-\dfrac{1}{\sqrt{(x-b)(x-c)}}\right\}$.

110. $1+x^{1/x-2}(1-\log x)$. 112. $\frac{1}{2}\sec x$.

CHAPTER III

Exercises XIII (Pages 58-59)

(1) 1. $\dfrac{3}{4}\sin\left(x+\dfrac{n\pi}{2}\right)-\dfrac{3^n}{4}\sin\left(3x+\dfrac{n\pi}{2}\right)$.

2. $\dfrac{4^n}{8}\cos\left(4x+\dfrac{n\pi}{2}\right)+2^{n-1}\cos\left(2x+\dfrac{n\pi}{2}\right)$.

3. $\dfrac{1}{2^7}\left\{8^n\sin\left(8x+\dfrac{n\pi}{2}\right)+2\cdot6^n\sin\left(6x+\dfrac{n\pi}{2}\right)\right.$

$\left.+4^{n+1}\sin\left(4x+\dfrac{n\pi}{2}\right)-6\cdot2^n\sin\left(2x+\dfrac{n\pi}{2}\right)\right\}$

4. $-\frac{1}{16}\left\{5^n \cos\left(5x+\frac{n\pi}{2}\right)+3^n \cos\left(3x+\frac{n\pi}{2}\right)\right.$
$\left.-2\cos\left(x+\frac{n\pi}{2}\right)\right\}.$

5. $\frac{1}{4}\left\{4^n \sin\left(4x+\frac{n\pi}{2}\right)+2^n \sin\left(2x+\frac{n\pi}{2}\right)\right.$
$\left.-6^n \sin\left(6x+\frac{n\pi}{2}\right)\right\}.$

6. $2^{n/2} e^x \sin\left(x+\frac{n\pi}{4}\right).$

7. $2^{2n-1} e^{4x}-\frac{1}{2}(20)^{n/2} e^{4x}\cos(2x+n\tan^{-1}\frac{1}{2}).$

8. $\frac{1}{2} e^x\left\{2^{n/2}\cos\left(x+\frac{n\pi}{4}\right)-10^{n/2}\cos(3x+n\tan^{-1}3)\right\}.$

9. $\frac{(-1)^{n-1}c^{n-1}(ad-bc)\,n\,!}{(cx+d)^{n+1}}.$

10. $\frac{3}{4} e^{5x}(25+a^2)^{n/2}\sin\left(ax+n\tan^{-1}\frac{a}{5}\right)$
$-\frac{1}{4}e^{5x}(25+9a^2)^{n/2}.\sin\left(3ax+n\tan^{-1}\frac{3a}{5}\right).$

11. $\frac{1}{4}\left[\frac{(-1)^n n\,!}{(x+\frac{1}{2})^{n+1}}-\frac{(-1)^n n\,!}{(x+\frac{3}{2})^{n+1}}\right].$

12. $\frac{4(-1)^n n\,!}{(x+2)^{n+1}}-\frac{3(-1)^n n\,!}{(x+1)^{n+1}}+\frac{(-1)^n(n+1)\,!}{(x+1)^{n+2}}.$

13. $\frac{16(-1)^n n\,!}{(x-2)^{n+1}}-\frac{(-1)^n n\,!}{(x-1)^{n+1}}.$

14. $\frac{(-1)^{n-1}(n-1)\,!}{(2+x)^n}-\frac{\underline{n-1}\,!}{(2-x)^n}.$

15. $\frac{(-1)^n n\,!}{2a}\left\{\frac{1}{(x-a)^{n+1}}-\frac{1}{(x+a)^{n+1}}\right\}.$

16. $(-1)^n n\,!\,.\,2^{n-1}\left\{\frac{1}{(2x-1)^{n+1}}-\frac{1}{(2x+1)^{n+1}}\right\}.$

17. $y_n=(-1)^n n\,!\,\Sigma\frac{a^2}{(a-b)(a-c)(x-a)^{n+1}}.$

18. $y_n=(-1)^n n\,!\,\Sigma\frac{a^3}{(a-b)(a-c)(x-a)^{n+1}}.$

(9) (i) $\dfrac{(-1)^{n-1}\,\overline{n-1}\,!}{2i}\left\{\dfrac{1}{(x-ai)^n}-\dfrac{1}{(x+ai)^n}\right\}.$

(ii) $\dfrac{(-1)^n\,n\,!}{2ib}\left\{\dfrac{1}{(x+a-ib)^{n+1}}-\dfrac{1}{(x+a+ib)^{n+1}}\right\}.$

(iii) $\dfrac{(-1)^n\,n\,!}{2\,(a^2-b^2)\,i}\left[\dfrac{1}{b}\left\{\dfrac{1}{(x-bi)^{n+1}}-\dfrac{1}{(x+bi)^{n+1}}\right\}\right.$

$$\left.-\frac{1}{a}\left\{\frac{1}{(x-ai)^{n+1}}-\frac{1}{(x+ai)^{n+1}}\right\}\right].$$

Exercises XIV (Pages 61-62)

16. $a=4$; $b=13$.

Exercises XV (Pages 64-65)

1. $e^x\,(n+x)$.
2. $e^{3x}\,(3^n\,x^2+3^{n-1}\,2nx+3^{n-2}\,n\,\overline{n-1})$.
3. $x\sin\left(x+\dfrac{n\pi}{2}\right)+n\sin\left(x+\overline{n-1}\,\dfrac{\pi}{2}\right)$.
4. $x^2\cos\left(x+\dfrac{n\pi}{2}\right)+2nx\cos\left(x+\dfrac{n-1}{2}\pi\right)$

$$+\,n\,(n-1)\cos\left(x+\frac{n-2}{2}\pi\right).$$

5. $e^x\left\{\log x+{}_nC_1\dfrac{1}{x}-{}_nC_2\dfrac{1}{x^2}+{}_nC_3\dfrac{1.2}{x^3}-{}_nC_4\dfrac{1.2.3}{x^4}\;......\right.$

$$\left.+\,(-1)^{n-1}\frac{\overline{n-1}\,!}{x^n}\right\}.$$

6. $x^3\left[\dfrac{3}{4}\sin\left(x+\dfrac{n\pi}{2}\right)-\dfrac{3^n}{4}\sin\left(3x+\dfrac{n\pi}{2}\right)\right]$

$$+3\,{}_1x^2\left[\frac{3}{4}\sin\left(x+\frac{n-1}{2}\pi\right)-\frac{3^{n-1}}{4}\sin\left(3x+\frac{n-1}{2}\pi\right)\right]$$

$$+\,3n\,(n-1)\,x\left[\frac{3}{4}\sin\left(x+\frac{n-2}{2}\pi\right)-\frac{3^{n-2}}{4}\sin\left(3x+\frac{n-2}{2}\pi\right)\right]$$

$$+\,n\,(n-1)\,(n-2)\left[\frac{3}{4}\sin\left(x+\frac{n-3}{2}\pi\right)\right.$$

$$\left.-\frac{3^{n-3}}{4}\sin\left(3x+\frac{n-3}{2}\pi\right)\right].$$

7. $a^x n! \left\{1 + {}_nc_1 \cdot \frac{x}{1!} \log a + {}_nc_2 \cdot \frac{x^2}{2!} (\log a)^2 + \ldots\ldots + \frac{x^n}{n!} (\log a)^n \right\}$.

CHAPTER IV

Exercises XVI (Pages 68-70)

2. x lies outside the range $(-2, 6)$.
3. $-2 < x < 7$. 12. 1 and -1. 13. $3b < a^2$.
14. a lies between ± 2.

Exercises XVII (Pages 74-77)

1. $3, \frac{1}{3}$. 3. $3\pi/2$ sq.ft. p.s. 4. 120π sq.ft. p.s.
5. $\frac{1}{90\pi}$ in. per sec. 6. 10. 7. 7/144. 8. 3/2 ft. p.s.; $10\sqrt{2}$.
9. $+5/3$ ft. p.s. towards O. 11. 6″ p.s.; 5″ p.s.
12. $3/8\pi$ cm./sec.
13. 1728π c.in./min. or π c.ft./min.; 48π sq.in./sec.
14. $1/\pi$ inch/min. 15. $1/\pi$ inch/sec. 16. $\frac{6}{25\pi}$ ft./sec.
17. $\frac{81}{8\pi}$ inch/min. 18. $\frac{2}{9\pi}$ ft./min. 19. $2\frac{1}{2}$ ft./hr.
20. $\frac{30}{7}$ m.p.h. 21. $\frac{350'}{3}$ per min. 22. $\frac{320}{343}$ sq.in./sec.
23. 5π sq.ft./sec. 24. $37\frac{1}{2}$ ft./sec. 25. 2·2 $(\sqrt{5})$ m.p.h.
26. $\frac{v\sqrt{x^2 - h^2}}{x}$ ft./sec. 27. $\frac{16}{17\sqrt{17}}$ radians per sec.
29. $\pi/2$ sq.ft./sec. 30. $\frac{120\pi ar^3}{(60r + at)^2}$ sq. miles per minute.

Exercises XVIII (Pages 80-81)

1. (1) $v = (20 - 4t)'$/sec.; $a = -4'$/sec.^{2} (2) $v = 3t^2 - 4t$; $a = 6t - 4$. (3) $u = -5\pi \sin \frac{\pi t}{2}$; $a = -\frac{5\pi^2}{2} \cos \frac{\pi t}{2}$.

(4) $v = \frac{6t^2 + 12t - 1}{(1 + t)^2}$; $a = \frac{14}{(1 + t)^3}$.

2. (1) Initial velocity $= 0$; $s = 4$.
   (2) Initial velocity $= 4$; $s = 8$.

3. (1) Velocity after 3 seconds $22\frac{1}{2}'$/sec., after 8 seconds, 0
Acceleration „ $3'$/sec.^{2} „ $-12'$/sec.^{2}
(2) When $t = 0$ or 8 seconds. (3) $24'$/sec.^{2}

4. (1) $16'$/sec. ; $-16'$/sec. (2) When $t = 1{\cdot}5$ seconds, $s = 100'$. (3) $80'$/sec.^{2} 5. $v = -3'$/sec. when acceleration $= 0$, $a = \pm 6'$/second^2 when velocity $= 0$.

7. $(x-a)^2 + (y-b)^2 = c^2$.

10. $-\sqrt{\dfrac{2a-x}{x}}$; $-\dfrac{a}{x^2}$. 13. $\dfrac{80\sqrt{2}}{3}$ ft./sec.

CHAPTER V

Exercises XX (Page 87)

5. $c = 3{\cdot}97$.

Exercises XXI (Page 89)

1. (1) 2, 2 and -5. (2) 1, 1, 1 and $\dfrac{-3 \pm i\sqrt{15}}{2}$.
(3) 3, 3, -2 and -2. (4) 1, 1 and -4.
(5) 3, 3 and $\pm i$. (6) $-\frac{1}{2}$, $-\frac{1}{2}$ and 5.

2. (1) Touches at $x = 2$ and cuts at $x = 3$ and -1.
(2) Touches at $x = 1$ and cuts at $x = 2$ and -1.
(3) Touches at $x = \frac{8}{3}$ and cuts at $\frac{4}{3}$.

3. 2, 2, 2 and $\dfrac{-9 \pm i\sqrt{15}}{8}$.

5. (1) $(-\infty, 0)$, $(0, 4)$, $(4, +\infty)$.
(2) $(-\infty, -3)$, $(-3, 0)$, $(0, 3)$, $(3, \infty)$.
(3) $(-\infty, 2)$, $(2, 6)$ and $(6, \infty)$.
(4) $(-\infty, 2)$, $(-2, 3)$ and $(3, \infty)$.
(5) $(-\infty, -5)$, $(-5, 0)$, $(0, 2)$ and $(2, \infty)$.

6. (1) Two roots real and two roots imaginary.
(2) One real negative root and two imaginary roots.
(3) One real negative root and two roots imaginary.

7. $27q^2 + 4p^3 = 0$.

Exercises XXII (Page 91)

1. 2·16. 2. 3·91. 3. 4·18. 4. 1·296. 5. 1·41. 6. 9·29.

Exercises XXIII (Pages 93-94)

1. -1. 2. 8/9. 3. $-1/6$. 4. 9/5. 5. $1/2\sqrt{2}$. 6. 2.
7. $-\infty$. 8. 1/2. 9. 1/2. 10. $1/\sqrt{2}$. 11. 1/2. 12. $\log(a/b)$.

13. $\cot^2 a$. 14. -1. 15. -2. 16. $2k$. 17. $a=-2$; limit $=-1$. 18. 1. 19. $\frac{1}{120}$.

Exercises XXIV (Page 96)

1. 5/3. 2. 3/2. 3. 0. 4. $\frac{1}{2}$. 5. 0. 6. 3. 7. 1. 8. 0. 9. 1. 10. ∞. 11. 0.

Exercises XXV (Page 98)

1. $1/\pi$. 2. $\pm 1/\pi$ according as n is even or odd. 3. $-1/a$. 4. 1. 5. 0. 6. $-1/2$. 7. 0. 8. 0. 9. 0. 10. 0. 11. 2/3. 12. -1. 13. 1/2. 14. ∞. 15. $2/\pi$.

Exercises XXVI (Page 100-1)

1. 1. 2. $e^{2/\pi}$. 3. $e^{-1/2}$. 4. $e^{-1/2}$. 5. e^2. 6. $1/e$. 7. $e^{-1/2}$. 8. $e^{2/\pi}$. 9. 1. 10. 1. 11. 1. 12. $e^{-x^2/2}$. 13. 1. 14. $e^{-1/6}$. 15. $e^{-1/2}$. 16. 1. 17. $\sqrt{ab}$.

CHAPTER VI

Exercises XXVII (Pages 108-10)

1. (*a*) Maximum at $x=1$, Maximum value $=11$
Minimum at $x=3$, Minimum value $=-17$.
(*b*) Maximum at $x=-5$, Maximum value $=0$
Minimum at $x=1$, Minimum value $=-324$.
(*c*) Minimum at $x=1$, Minimum value $=-1$.
(*d*) Maximum at $x=4$, Maximum value $=164$
Minimum at $x=8$, Minimum value $=132$.
(*e*) Maximum at $x=-4$, Maximum value $=-6$
Minimum at $x=0$, Minimum value $=2$.
(*f*) Maximum at $x=17/7$, Maximum value $=\left(\frac{3}{7}\right)^3\left(\frac{4}{7}\right)^4$.
(*g*) Maximum at $x=1$; Maximum value $=\frac{1}{2}$
Minimum at $x=-1$; Minimum value $=-\frac{1}{2}$.
(*h*) Minimum 25 at $x=16$; Maximum 1 at $x=4$.
(*i*) Turning values at $x=1\pm\sqrt{3}$ and $x=1$;
Max. $y=0$ at $x=1$.

2. Max. is at $x=\cos^{-1}\frac{3}{5}$ and its value is $8\left(\frac{4}{5}\right)^5$.

3. Max. at $x = -1$, and its value 3. Min. at $x = 1$, and its value $\frac{1}{2}$.

4. Max. radius vector is $\dfrac{c}{a+b}$. Min. radius vector is zero.

5. $-e^{-1}$. 7. e^{-1}.

Exercises XXVIII (Pages 110-2)

2. $c = t/r$. 4. $2' 1''$ (approximately). 5. (*a*) $x = \sqrt{nR/r}$; (*b*) 6. 7. t is given by the equation $t^2 + 2\mu t - 1 = 0$. 9. $5' 6{\cdot}6''$ nearly from the 16 power candle. 11. $\alpha = 38°$ and $E = 0{\cdot}7$ nearly. 12. $2x^3 - 3x^2/2 - 9x$.

Exercises XXIX (Pages 117-21)

3. $\dfrac{l^2}{8+2\pi}$ sq. units. 8. $\dfrac{3\sqrt{3}ab}{4}$ sq. units. 14. Base radius $= 1'$; semi-vertical angle $\sin^{-1}\left(\dfrac{1}{3}\right)$. 22. P is at a distance $ca^{3/2}(a^{3/2} + b^{3/2})^{-1}$ from the centre of the sphere of radius a. 26. $13{\cdot}9$. 27. Breadth $2a/\sqrt{3}$; depth $= 2a\sqrt{\frac{2}{3}}$. 28. $3:6:4$. 34. $1/4$. 37. $3a\sqrt{3}/2$; $8a^2/3\sqrt{3}$ where $2a$ is the breadth of the sheet.

Exercises XXX (Pages 123-4)

1. (1) $(1, 2)$ and $(-1, -14)$. (2) $(3, -39)$.

(3) $\left\{(4n+1)\frac{\pi}{2}, 0\right\}$; n is any +ve integer.

(4) $\left\{2n\pi \pm \frac{\pi}{2} + \tan^{-1}\frac{a}{b}, 0\right\}$; n is a +ve integer.

(5) $\left\{n\pi \pm \frac{\pi}{4}, \frac{a+b}{2}\right\}$. (6) $\left\{0, \pm\sqrt{\frac{3}{5}}a\right\}$.

(7) $(0, 0)$, $\left\{\sqrt{\frac{3}{5}}a, \frac{3}{8}\sqrt{\frac{3}{5}}a\right\}$

$\left\{-\sqrt{\frac{3}{5}}a, -\frac{3}{8}\sqrt{\frac{3}{5}}a\right\}$.

(8) $\{(3+\sqrt{3});\ (3+\sqrt{3})^3 e^{-3-\sqrt{3}}\}$;

$\{(3-\sqrt{3});\ (3-\sqrt{3})^3 e^{-3+\sqrt{3}}\}$.

(9) $\left\{e^{15/4}, \frac{15}{4}e^{-5/4}\right\}$.

3. At $x = \dfrac{a+b+c}{3}$.

7. $3a + b = 0$ for all values of c and d. 9. $8a/27b$.

CHAPTER VII

Exercises XXXI (Pages 134-5)

1. (1) $x+\frac{x^3}{3!}+16\frac{x^5}{5!}+272\frac{x^7}{7!}+\ldots$

(2) $1+x-\frac{2x^3}{3!}-\frac{4x^4}{4!}-\frac{4x^5}{5!}+\frac{8x^7}{7!}+\ldots$

(3) $1+\frac{x^2}{2!}+\frac{5x^4}{4!}+61\frac{x^6}{6!}+\ldots$

(4) $1+x+\frac{2x^2}{2!}+\frac{4x^3}{3!}+\frac{12x^4}{4!}+\frac{36x^5}{5!}+\ldots$

(5) $\log 2+\frac{1}{2}x+\frac{1}{4}\frac{x^2}{2!}-\frac{1}{8}\frac{x^4}{4!}+\ldots$

(6) $1+x+\frac{x^2}{2!}-5\frac{x^4}{4!}+\ldots$

(7) $x+\frac{2x^2}{2!}+\frac{2x^3}{3!}-\frac{4x^5}{5!}+\ldots$

(8) $1-\frac{2^2x^4}{4!}+\frac{2^4x^8}{8!}-\frac{2^6x^{12}}{12!}+\ldots$

$$\ldots+\frac{(-1)^n\,.\,2^{2n}\,.\,x^{4n}}{4n\,!}+\ldots$$

(9) $-x-\frac{2x^3}{3!}+\frac{4x^3}{5!}+\frac{8x^7}{7!}-\frac{16x^9}{9!}-\frac{32x^{11}}{11!}+\ldots$

(10) $\frac{x^2}{2!}+\frac{x^4}{4!}+16\frac{x^6}{6!}+272\frac{x^8}{8!}+\ldots$

9. $\frac{1}{\sqrt{2}}\left[1+\theta-\frac{\theta^2}{2!}-\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+\frac{\theta^5}{5!}+\ldots\right]$.

CHAPTER VIII

Exercises XXXII (Pages 140-2)

12. 0. 16. 0.

Exercises XXXIII (Pages 150-2)

1. (1) $-\frac{\tan t\,(1+2\cos t)}{1+\cot t}$. (2) $e^{-2t}\sin^3 t\,(3\cot t-2)$.

(3) $26t^{25}$. (4) $y^2\left(\frac{1}{t}+2x\right)\cos(xy^2)$.

2. (1) $\frac{2x\,(y-x)}{y}$. (2) $\frac{\tan x\,(2x\sec^2 x-\tan x)}{x^2+y^2}$.

(3) $\dfrac{x^2y\,(4x^2+xy-6y^2)}{x-2y}$. (4) $\log(xy)+2-\dfrac{ax}{y^2-ax}$.

(5) $2e^2x\cos(x^2+y^2)$, where $b^2=a^2(1-e^2)$.

3. (1) $\dfrac{2a}{y}$. (2) $-\dfrac{b^2x}{a^2y}$. (3) $-\dfrac{y}{x}$.

(4) $-\dfrac{ax+hy+g}{hx+by+f}$. (5) $-\dfrac{x-a}{y-b}$.

Exercises XXXIV (Pages 160-5)

1. $\dfrac{\pi}{2}$. 2. 0·7. 3. $\dfrac{5\pi}{36}\cot 63^\circ$. 6. $\dfrac{2\pi}{3}$. 10. $\dfrac{50}{81}$. 11. $-\frac{1}{2}$.

13. 0·41 sq. in. 14. $\dfrac{e\sin x}{2\sin y}(2\cot x-\cot y)$. 15. $\dfrac{\pi\sqrt{3}}{81}$.

16. $-0{\cdot}96\pi c$ in. 17. $(a+b+c)$. 18. 0·006″ (approx.).

21. 0·037 ohm ; 0·6 %. 22. 3·5 %. 23. $(2+a\cot 2a)$ %.

25. 0·0080 ; 0·004 (approx.). 26. 3 %. 35. $2p$.

CHAPTER IX

Exercises XXXV (Pages 171-4)

1. (1) $3x-4y-1=0$; $32x+24y-19=0$.

(2) $3x+5y-14=0$; $5x-3y-12=0$.

(3) $ax\cos x_1-y\sin x_1=ax_1\cos x_1-y_1\sin x_1$

$x\sin x_1+ay\cos x_1=x_1\sin x_1+ay_1\cos x_1$.

(4) $4x+2y-3=0$; $2x-4y+1=0$.

(5) $\dfrac{x}{a}\operatorname{cosec} t+\dfrac{y}{b}\sec t=1$

$ax\sin t-by\cos t=a^2\sin^4 t-b^2\cos^4 t$.

3. $\dfrac{x}{a}\cos\theta+\dfrac{y}{b}\sin\theta=1$; $\dfrac{ax}{\cos\theta}-\dfrac{by}{\sin\theta}=a^2-b^2$.

4. $2x+y=2$; $x+2y=6$. 5. Tangent is $y=0$.

7. $(2\sqrt{2},\,5\sqrt{2})$ and $(-2\sqrt{2},\,-5\sqrt{2})$.

8. Tangent at $\left(\frac{4}{3}a,\ \frac{2a}{3}\sqrt[3]{4}\right)$ is parallel to the x-axis and at $(2a, 0)$ is parallel to the y-axis.

10. $x(4h^3+a^2k)+y(a^2h+b^3)=4h^4+2a^2hk+b^3k$.

11. $3x+y=10$. 12. $4x-y=1$. 13. $y=x+1$.

17. $3tx=2y+t^3$, where P is (t^2, t^3).

18. $y = xt - 2at - at^3$. 19. $\left(\frac{4}{3}a, \frac{2a}{3}\sqrt[3]{4}\right)$.

22. $2x - y = a$. 23. Tangent is $3mx = 2y + am^3$, $m + 2m' = 0$, where m' is the parameter of the point Q.

27. $76x - 48y = 153$. 29. $4a/9$.

Exercises XXXVI (Pages 176-8)

1. $\frac{x}{\cos^2\theta} + \frac{y}{\sin^2\theta} = a$.

4. $4x \sin^3\theta = 4y \cos^3\theta - 3\sin 4\theta$.

5. $x = y \tan\frac{\theta}{2} + a\theta$. 6. $x = \cot\frac{\theta}{2} y + a\theta$.

9. $\frac{x}{a}\sec\theta + \frac{y}{a}\operatorname{cosec}\theta = 1$.

11. $2(x^2 + y^2)^3 = a^2(x^2 - y^2)^2$.

12. $n = 1$. 14. $3y + at^4 = 4tx$.

Exercises XXXVII (Pages 180-1)

1. $\tan^{-1}(3)$. 2. $\tan^{-1}(6/13)$. 3. $\tan^{-1}(\frac{1}{3})$.

4. $\tan^{-1}\left(\frac{11}{23}\right)$. 6. 45^c.

Exercises XXXVIII (Pages 182-3)

1. S.T. $= \frac{a}{2}$; T $= \frac{a}{2}\sqrt{5}$; S.N. $= 2a$; N $= a\sqrt{5}$.

3. $a \tan\frac{x}{a}$; $\frac{b^2}{2a}\sin\frac{2x}{a}$. 5. $\tan x$; $\frac{1}{2}\sin 2x$.

9. $a\sin\theta$; $a(1 - \cos\theta)\tan\frac{\theta}{2}$.

10. $a(\sin t - t\cos t)\cdot\cot t$; $a(\sin t - t\cot t)\tan t$.

13. a. 17. $n = -2$.

Exercises XXXIX (Page 185)

2. (1) $\left(\frac{a}{x}\right)^{1/3}$; (2) $\sqrt{\frac{x+y}{x}}$; (3) $\cosh\frac{x}{a}$.

Exercises XL (Pages 193-4)

1. (1) -1; (2) -1; (3) 1; (4) 0; (5) $-\pi/2$.

2. $\frac{1}{2}(\pi + 4\theta)$. 6. (*i*) $\tan^{-1}\frac{4}{3}$; (*ii*) $\frac{\pi}{2}$.

8. S.T. $= \dfrac{a^\theta}{\log a}$; S.N. $= a^\theta \log a$;

$T = \dfrac{a^\theta}{\log a}\sqrt{1 + (\log a)^2}$; $N = a^\theta \sqrt{1 + (\log a)^2}$.

12. $\phi = \dfrac{\pi}{2} - (n\theta + a)$.

16. (*i*) $\sqrt{r^2 + 9\cot^2 3\theta}$; (*ii*) $r\sqrt{8r - 3}$.

CHAPTER X

Exercises XLI (Pages 200-3)

1. (*a*) $2\sqrt{2}$; (*b*) $\dfrac{1}{\sqrt{2}}$; (*c*) 6; (*d*) $\dfrac{(109)^{3/2}}{.60}$; (*e*) $\dfrac{a}{3}$;
(*f*) $\dfrac{10^{3/2}\,a}{12}$; (*g*) $\dfrac{3a}{8\sqrt{2}}$; (*h*) $\dfrac{5^{3/2}}{4}$; (*i*) $2\sqrt{2}/3$.

2. $\dfrac{(a^2 + b^2)^{3/2}}{ab}$. 3. $2a\,(1 + t^2)^{3/2}$.

9. $\dfrac{\{x^{2(m-1)} + y^{2(m-1)}\}^{3/2}}{(m-1)\,x^{m-2}\,y^{m-2}}$.

11. $\{a\,(2^{2/3} - 1),\ 2a\,(2^{2/3} - 1)^{1/2}\}$.

12. 125/64. 13. (1) $\tan^{-1}\dfrac{3}{4}$; (2) $\dfrac{1}{5\sqrt{5}}$.

15. $a \tan\theta \sec\theta$. 19. $\left(-\dfrac{1}{2}\log 2, \dfrac{1}{\sqrt{2}}\right)$.

20. $(1, \pm 2)$. 22. $\frac{1}{3}$. 24. $\dfrac{(t^2 - 1)^2}{4t^2}$.

Exercises XLII (Page 207)

1. (1) $\left(-\dfrac{1}{2}, \dfrac{5}{4}\right)$; (2) $(2c, 2c)$; (3) $\left\{\dfrac{\pi}{2}\sqrt{3}, \log 2c\right\}$.

2. $(2 + 3t^2, -2t^3)$.

3. $x = 2\,(3\cos t - \cos 3t)$;
$y = 2\,(3\sin t + \sin 3t)$.
When $t = 0$, the centre of curvature is (4, 0).

Exercises XLIII (Pages 213-4)

1. (1) $r\sec 2\theta$. (2) $\dfrac{(r^2e^2 + 2rl - r^2)^{3/2}}{l^2}$.

(3) $2a\sec^3\dfrac{\theta}{2}$. (4) $\dfrac{a\,(1 + \theta^2)^{3/2}}{(2 + \theta^2)}$. (5) $r\sqrt{1 + \cos^2 a}$.

3. $\dfrac{a^n r^{1-n}}{1+n}$. 8. (1) $p + a = 0$. (2) $ap^2 = r^3$. (3) $ap = r^2$. (4) $pr + a^2 = 0$. (5) $pa^m = r^{m+1}$. (6) $p = r \sin a$.

CHAPTER XII

Exercises XLVI (Page 237)

1. $-x^{-3}/3$.
2. $2x^{5/2}/5$.
3. $\dfrac{ax^2}{2} - \dfrac{b}{x}$.
4. $a \log x - \dfrac{b}{x} - \dfrac{c}{2x^2}$.
5. $\dfrac{ax^3}{3} + \dfrac{bx^4}{4} + \dfrac{cx^5}{5}$.
6. $\dfrac{x^3}{3} + 2x = \dfrac{1}{x}$.
7. $\dfrac{5}{9}x^{9/5} - \dfrac{5}{2}x^{4/5} - 5x^{-1/5}$.
8. $\dfrac{x^3}{3} - \dfrac{x^4}{2} + \dfrac{x^5}{5}$.
9. $\dfrac{x^3}{3} + 2x^2 + 6x + 4 \log x - \dfrac{1}{x}$.
10. $\log x - x^2 + \dfrac{x^4}{4}$.
11. $\dfrac{x^5}{5} - \dfrac{5}{6}x^{12/5} - 5x^{-1/5}$.
12. $2x^{1/2} - 6x^{-1/2}$.
13. $\dfrac{6}{5}x^{5/2} + \dfrac{8}{3}x^{3/2} - 10x^{1/2}$.
14. $2x + 5 \log x + \dfrac{12}{x}$.
15. $\tan x - x$.
16. $-\cot x - x$.
17. $\tan x - 4 \cot x - 9x$.
18. $\tan x - \cot x$.
19. $x - \sin x$.
20. $x - \cos x$.
21. $3 \sin^{-1} x + e^x + 8x$.
22. $\sin x - \cos x$.
23. $\tan x - \sec x$.
24. $\tan x + \sec x$.
25. $\operatorname{cosec} x - \cot x$.
26. $-(\cos x + \operatorname{cosec} x)$.

Exercises XLVII (Pages 242-3)

1. $x^4/4$, $(2+x)^4/4$, $-(3-2x)^4/8$, $(3x-4)^4/12$, $(ax+b)^4/4a$, $-(b-ax)^4/4a$.

2. $\dfrac{x^{n+1}}{n+1}$, $\dfrac{(x+a)^{n+1}}{n+1}$, $\dfrac{(3x+2)^{n+1}}{3(n+1)}$, $-\dfrac{(3-2x)^{n+1}}{2(n+1)}$, $\dfrac{(ax-b)^n}{(n+1)a}$, $-\dfrac{(b-ax)^{n+1}}{(n+1)a}$.

3. $\dfrac{2x^{3/2}}{3}$, $\dfrac{2}{3}(1+x)^{3/2}$, $\dfrac{2}{9}(2+3x)^{3/2}$, $\dfrac{-2}{15}(4-5x)^{3/2}$, $\dfrac{2}{3b}(a+bx)^{3/2}$, $\dfrac{2c}{3b}\left(a+\dfrac{bx}{c}\right)^{3/2}$.

4. $-\dfrac{1}{3x^2}$, $-\dfrac{1}{3(x+7)^3}$, $-\dfrac{1}{6(2x+3)^3}$, $\dfrac{1}{9(4-3x)^3}$,
$-\dfrac{1}{3b}\dfrac{1}{(a+bx)^3}$, $\dfrac{1}{3b(a-bx)^3}$.

5. $-\dfrac{2}{\sqrt{x}}$, $-\dfrac{1}{\sqrt{2x+1}}$, $-\dfrac{2}{3}\dfrac{1}{\sqrt{3x-4}}$, $\dfrac{2}{3}\dfrac{1}{\sqrt{2-3x}}$,
$-\dfrac{2}{b\sqrt{a+bx}}$, $\dfrac{2}{b}\dfrac{1}{\sqrt{a+bx}}$.

6. $\dfrac{\log x}{2}$, $\dfrac{\log x}{a}$, $\dfrac{\log(3x+7)}{3}$, $-\dfrac{\log(2-7x)}{7}$,
$\dfrac{\log(ax+b)}{a}$, $-\dfrac{\log(b-ax)}{a}$.

7. $\dfrac{e^{4x}}{4}$, $\dfrac{e^{3x+7}}{3}$, $-\dfrac{e^{2-3x}}{3}$, $3\,e^{(x-4)/3}$, $\dfrac{1}{a}e^{ax+b}$.

8. $-\dfrac{\cos 2x}{2}$, $-2\cos\dfrac{x}{3}$, $-\dfrac{\cos(2x+3)}{2}$, $\dfrac{\cos(3-2x)}{2}$.

9. $\dfrac{\sin 3x}{3}$, $3\sin\dfrac{x}{3}$, $\dfrac{\sin(3x+2)}{3}$, $\dfrac{-\sin(2-3x)}{3}$.

10. $\dfrac{\tan 4x}{4}$, $4\tan\dfrac{x}{4}$, $\dfrac{\tan(4x-7)}{4}$, $\dfrac{-\tan(7-4x)}{4}$.

11. $\dfrac{-\cot 5x}{5}$, $-5\cot\dfrac{x}{5}$, $\dfrac{-\cot(5x+3)}{5}$, $\dfrac{\cot(3-5x)}{5}$.

12. $\dfrac{\sec 2x}{2}$, $2\sec\dfrac{x}{2}$, $\dfrac{\sec(2x-3)}{2}$.

13. $\dfrac{-\operatorname{cosec} 3x}{3}$, $-3\operatorname{cosec}\dfrac{x}{3}$, $\dfrac{\operatorname{cosec}(b-ax)}{a}$.

14. $\dfrac{2(x+a)^{3/2}(3x-2a)}{15}$.

15. $\dfrac{2}{3b^2}\sqrt{a+bx}\,(bx-2a)$.

16. $\dfrac{3}{10b^2}(2bx-3a)(a+bx)^{2/3}$.

17. $\dfrac{2}{3a}[(x+a)^{3/2}-x^{3/2}]$.

18. $\dfrac{[bx(1-n)-a]}{b^2(n-1)(n-2)(a+bx)^{n-1}}$.

19. $\dfrac{b}{b_1}x+\dfrac{ab_1-ba_1}{b_1^2}\log(a_1+b_1x)$.

20. & 21. $2\tan\dfrac{x}{2}-x$. 22. $x-\tan\dfrac{x}{2}$.

23. $-\frac{1}{2}\left(\frac{\cos 5x}{5}+\cos x\right)$. 24. $\frac{1}{2}\left(-\frac{\sin 9x}{9}+\frac{\sin 3x}{3}\right)$.

25. $\frac{1}{2}\left(\frac{\sin 9x}{9}+\sin x\right)$.

27. $\frac{1}{2}\left[x-\frac{\sin(4x+10)}{4}\right]$.

28. $\frac{1}{8}\left(-\cos 2x+\frac{\cos 6x}{3}-\frac{\cos 4x}{4}\right)$.

29. $\frac{1}{4}\left(\frac{\sin 12x}{12}+\frac{\sin 8x}{8}+\frac{\sin 4x}{4}+x\right)$. 30. $\frac{1}{8}\left(x-\frac{\sin 4x}{4}\right)$.

31. $-\frac{3}{4}\cos x+\frac{1}{12}\cos 3x$.

32. $\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x$.

33. $\frac{1}{4}\left(\frac{2\sin 3x}{3}-\frac{\sin 5x}{5}-\sin x\right)$.

34. $\frac{1}{64}\left(-3\cos 2x+\frac{\cos 6x}{3}\right)$.

35. $\frac{\tan 3x}{3}-x$. 36. $\frac{3\pi}{32}-\frac{1}{4}$.

Exercises XLVIII (Pages 244-5)

1. $\frac{1}{2}\tan^{-1}2x$, $2\tan^{-1}\frac{x}{2}$, $\frac{1}{\sqrt{7}}\tan^{-1}\sqrt{7}\,x$,

$\frac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}\,(x+2)$, $\frac{1}{ab}\tan^{-1}\frac{bx}{a}$.

2. $\frac{1}{\sqrt{3}}\tan^{-1}\frac{x+2}{\sqrt{3}}$, $\frac{1}{4}\tan^{-1}\frac{x-2}{4}$, $\frac{1}{12}\tan^{-1}\frac{3x+2}{4}$, $\frac{1}{6}\tan^{-1}\frac{2x+3}{3}$.

3. $\frac{1}{4}\log\frac{1+2x}{1-2x}$, $\log\frac{2+x}{2-x}$, $\frac{1}{2\sqrt{7}}\log\frac{1+\sqrt{7}\,x}{1-\sqrt{7}\,x}$,

$\frac{1}{2\sqrt{2}}\log\frac{1+\sqrt{2}\,(x+2)}{1-\sqrt{2}\,(x+2)}$, $\frac{1}{2ab}\log\frac{a+bx}{a-bx}$.

4. $\frac{1}{12}\log\frac{3x-2}{3x+2}$, $\frac{1}{12}\log\frac{x-6}{x+6}$, $\frac{1}{12}\log\frac{2x-3}{2x+3}$,

$\frac{1}{2\sqrt{ab}}\log\frac{\sqrt{b}\,x-\sqrt{a}}{\sqrt{b}\,x+\sqrt{a}}$, $\frac{1}{2b\sqrt{a}}\log\frac{\sqrt{a}\,x-b}{\sqrt{a}\,x+b}$.

5. $\sin^{-1}\dfrac{x}{2}, \dfrac{1}{2}\sin^{-1}2x, \dfrac{1}{5}\sin^{-1}\dfrac{5x}{3}, \dfrac{1}{b}\sin^{-1}\dfrac{bx}{a}, \dfrac{1}{\sqrt{b}}\sin^{-1}\dfrac{\sqrt{b}}{\sqrt{a}}x.$

6. $\cosh^{-1}x, \dfrac{1}{2}\cosh^{-1}2x, \dfrac{1}{b}\cosh^{-1}\dfrac{xb}{a}, \cosh^{-1}\dfrac{x-2}{5}.$

7. $\sinh^{-1}\dfrac{x}{2}, \dfrac{1}{2}\sinh^{-1}2x, \dfrac{1}{5}\sinh^{-1}\dfrac{5x}{3}, \dfrac{1}{b}\sinh^{-1}\dfrac{bx}{a}, \dfrac{1}{2}\sinh^{-1}\dfrac{2x-3}{\sqrt{7}}.$

Exercises XLIX (Pages 245-6)

1. $-\dfrac{\cos(x^n)}{n}.$ 2. $-2\cos\sqrt{x}.$ 3. $\dfrac{1}{3a^3}\tan^{-1}\left(\dfrac{x}{a}\right)^3.$

4. $\dfrac{1}{6}\log\dfrac{1+x^3}{1-x^3}.$ 5. $\dfrac{1}{3}\sinh^{-1}x^3.$ 6. $\dfrac{1}{2a^2}\tan^{-1}\left(\dfrac{x}{a}\right)^2.$

7. $\dfrac{1}{2}\sin^{-1}x^2.$ 8. $\dfrac{1}{4}\sin^{-1}\left(\dfrac{x}{a}\right)^4.$ 9. $-\dfrac{2}{3}\sqrt{a^3-x^3}.$

10. $\dfrac{1}{3}\tan^{-1}x^3.$ 11. $\dfrac{1}{3}\sin^{-1}x^3.$ 12. $\dfrac{x^2-a^2\log(x^2+a^2)}{2}.$

13. $\dfrac{1}{4}\log(x^4-9)+\dfrac{1}{12}\log\dfrac{x^2-3}{x^2+3}.$

14. $\sqrt{x^2+1}+\sinh^{-1}x.$ 15. $\log(x^4+1)+\frac{3}{2}\tan^{-1}x^2.$

16. $\dfrac{5}{2}\log 2+\dfrac{\pi}{8}.$ 17. $\dfrac{1}{2}.$

Exercises L (Pages 247-8)

1. $\dfrac{(x^2+1)^{9/2}}{9}.$ 2. $-\dfrac{1}{\sqrt{1+x^2}}.$ 3. $-\dfrac{5(1-2x^2)^{3/2}}{6}.$

4. $\dfrac{1}{2(1-x^2)}.$ 5. $\dfrac{(a^2+x^2)^{5/2}}{5}.$ 6. $\dfrac{2(a^3+x^3)^{3/2}}{9}.$

7. $-\dfrac{(a-bx^2)^{n+1}}{2b(n+1)}.$ 8. $\dfrac{1}{6}\log(a^6+x^6).$

9. $\dfrac{(e^x+k)^{n+1}}{n+1}.$ 10. $-\dfrac{\log(1-x^3)}{3}.$ 11. $\dfrac{(\log x)^2}{2}.$

12. $\log\log x.$ 13. $\dfrac{1}{1-n}\dfrac{1}{(\log x)^{n-1}}.$ 14. $\dfrac{(1+\log x)^{n+1}}{n+1}.$

15. $\dfrac{2(1+\sqrt{x})^{n+1}}{n+1}.$ 16. $-\dfrac{1}{x+\sin x}.$

17. $-\dfrac{1}{6(3x^2+4x+2)^3}.$ 18. $\dfrac{\tan^{n+1}x}{n+1}.$

19. $-\dfrac{(1-\tan x)^{n+1}}{n+1}.$ 20. $-\dfrac{\cot^{n+1}x}{n+1}.$

21. $+\dfrac{1}{(n-1)\cos^{n-1}x}$. 22. $\dfrac{\sec^3 x}{3}$. 23. $\dfrac{1}{1+\cos x}$.
24. $\log(e^x+1)$. 25. $\log(e^x+e^{-x})$. 26. $\log\tan^{-1}x$.
27. $\log\sin^{-1}x$. 28. $\log(x+\sin x)$. 29. $\frac{1}{2}\log(e^{2x}-1)$.
30. $\dfrac{1}{bn}\log(a+bx^n)$. 31. $\dfrac{1}{b}\log(a+b\sin x)$.
32. $-\dfrac{1}{b}\log(a+b\cos x)$. 33. $\log\log\sin x$.
34. $-\log\log\cos x$. 35. $\frac{1}{4}\log(5+4\tan x)$.
36. $\dfrac{1}{3}\log(x^3-6x+4)$. 37. $\left(\dfrac{1}{a}-\dfrac{1}{x}\right)^{n+1}\Big/(n+1)$.
38. $-\dfrac{1}{2(2e^x-3)}$. 39. $\log\sin x-\cos x$.
40. $\frac{1}{4}\log(\sec 4x+\tan 4x)$. 41. $2\sin x-\log(\sec x+\tan x)$.
42. $\frac{5}{8}-2\log 2$. 43. $\log 2$. 44. $\tan^m x/m$.

Exercises LI (Pages 250-2)

1. $-\cos\log x$. 2. $\tan\log x$. 3. $e^{\tan^{-1}x}$.
4. $-\cos(\tan^{-1}x)$. 5. $-\dfrac{\tan^{-1}(3\cos x)}{3}$.
6. $\dfrac{1}{2}\tan^{-1}\dfrac{\sin x}{2}$. 7. $e^{\sin^2 x+\cos x}$. 8. $\cos^{-1}\dfrac{\cos x}{\sqrt{2}}$.
9. $-\sqrt{1-x^2}+2\sin^{-1}x$. 10. $-1/\sin^{-1}x$.
11. $\dfrac{\sin^5 x}{5}-\dfrac{\sin^7 x}{7}$. 12. $2\sqrt{\sin x}\left(1-\dfrac{\sin^2 x}{5}\right)$.
13. $\sec x\left(\dfrac{\sec^2 x}{3}-1\right)$. 14. $-\operatorname{cosec} x-2\sin x+\dfrac{\sin^3 x}{3}$.
15. $2\sin^{7/2}x\left(\dfrac{1}{7}-\dfrac{\sin^2 x}{11}\right)$. 16. $\dfrac{\tan^9 x}{9}$.
17. $\log y-2y^2+\dfrac{3y^4}{2}-\dfrac{2y^6}{3}+\dfrac{y^8}{8}$ where $y=\sin x$.
18. $2\cos^{3/2}\theta\left(-\dfrac{1}{3}+\dfrac{\cos^2\theta}{7}\right)$.
19. $\cos x\left(-1+\dfrac{2\cos^2 x}{3}-\dfrac{\cos^4 x}{5}\right)$.
20. $\sin x\left(1-\dfrac{2\sin^2 x}{3}+\dfrac{\sin^4 x}{5}\right)$.
21. $-y+y^3-\dfrac{3y^5}{5}+\dfrac{y^7}{7}$ where $y=\cos x$.

22. $\tan x + \dfrac{\tan^3 x}{3}$. 23. $\dfrac{\tan^3 x}{3} - \tan x + x$.

24. $-\dfrac{\cot^3 x}{3} + \cot x + x$. 25. $\log \cos x + \dfrac{\sec^2 x}{2}$.

26. $\dfrac{\sec^4 x}{4} - \sec^2 x - \log \cos x$.

27. $\log \sin x + \operatorname{cosec}^2 x - \dfrac{\operatorname{cosec}^4 x}{4}$. 28. $\log \tan x$.

29. $-2 \cot 2x$. 30. $\dfrac{1}{2(b^2 - a^2)} \log (a^2 \cos^2 x + b^2 \sin^2 x)$.

31. $-\frac{1}{3} \log (2 + 3 \cos^2 x)$. 32. $\log (1 + e^x)$.

33. $-\log (1 + e^{-x})$. 34. $\dfrac{7}{10} x + \dfrac{1}{10} \log (3 \sin x + \cos x)$.

35. $2\theta + \log (2 \cos \theta + \sin \theta)$. 36. $-\dfrac{a}{b} \log (be^{-x} + c)$.

37. $-\dfrac{1}{n} \dfrac{1}{(\sec x + \tan x)^n}$. 38. $\dfrac{\tan x + n \sec x}{(1 - n^2)(\sec x + \tan x)^n}$.

45. $\dfrac{8}{105} - \dfrac{11\sqrt{2}}{220}$. 46. $\dfrac{2}{15}$. 47. $\dfrac{1}{2}\left(\log 2 - \dfrac{1}{2}\right)$.

48. $\tan^{-1} e - \dfrac{\pi}{4}$. 49. $\dfrac{1}{ab} \cot^{-1} \dfrac{a}{b}$. 50. $\dfrac{\pi}{2ab}$. 51. $\dfrac{64}{4641}$.

Exercises LII (Page 253)

1. $\dfrac{x^2}{2} + 3 \log (x - 1)$. 2. $\dfrac{x}{2} - \dfrac{3}{4} \log (3 + 2x)$.

3. $\dfrac{lx}{a} + \dfrac{am - lb}{a^2} \log (ax + b)$. 4. $\dfrac{x(x-1)}{4} + \dfrac{1}{8} \log (2x + 1)$.

5. $-\left(x + \dfrac{x^2}{2} + \dfrac{x^3}{3}\right) - \log (1 - x)$.

6. $\dfrac{x^3}{3} - 3x + 3\sqrt{3} \tan^{-1} \dfrac{x}{\sqrt{3}}$.

7. $x + \log (x^2 + 1) + 4 \tan^{-1} x$.

8. $\dfrac{x^2}{2} - \dfrac{1}{2} \log (x^2 + 1) + \tan^{-1} x$. 9. $\dfrac{1}{3}(x^3 - \tan^{-1} x^3)$.

10. $\dfrac{1}{7}\left[\dfrac{x^{14}}{2} - 2 \log (x^{14} + 4)\right]$. 11. $\dfrac{x^3}{3} + 2x + \dfrac{3}{2} \log \dfrac{x-1}{x+1}$.

12. $-\log (1 - \sin x) + \dfrac{(1 - \sin x)(3 + \sin x)}{2}$.

13. $\dfrac{1}{a^2} [\log (1 - a \cos x) - (1 - a \cos x)]$.

Exercises LIII (Page 256)

1. $\frac{1}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}}$. 2. $\frac{1}{\sqrt{41}}\log\frac{\sqrt{41}+2x-5}{\sqrt{41}-2x+5}$.

3. $\frac{3}{2}\log(x^2+4x+7)-\frac{1}{\sqrt{3}}\tan^{-1}\frac{x+2}{\sqrt{3}}$.

4. $\log(x^2+2x+5)+\frac{1}{2}\tan^{-1}\frac{x+1}{2}$.

5. $\log(x^2+21x+3)-\frac{20}{\sqrt{429}}\log\frac{2x+21-\sqrt{429}}{2x+21+\sqrt{429}}$.

6. $\frac{3}{4}\log(2x^2-x+5)+\frac{7}{2\sqrt{39}}\tan^{-1}\frac{4x-1}{\sqrt{39}}$.

7. $-\frac{1}{2}\log(-1+x-2x^2)-\sqrt{7}\tan^{-1}\frac{4x-1}{\sqrt{7}}$.

8. $\frac{5}{12}\log(6x^2+4x-1)+\frac{8}{3\sqrt{10}}\log\frac{3\sqrt{2}x+\sqrt{2}-\sqrt{5}}{3\sqrt{2}x+\sqrt{2}-\sqrt{5}}$.

9. $\frac{3}{4}\log(2x^2+x+3)+\frac{1}{2\sqrt{23}}\tan^{-1}\frac{4x+1}{\sqrt{23}}$.

10. $\frac{5}{2}\log(x^2-2x-35)+\frac{1}{2}\log\frac{x-7}{x+5}$.

11. $\frac{1}{17}\log\frac{3x-2}{3(x+5)}$.

12. $\log\frac{e^x-2}{e^x-1}$. 13. $\frac{1}{3}\tan^{-1}(1+x^3)$.

14. $\frac{1}{8\sqrt{5}}\log\frac{\sqrt{5}+x^4-1}{\sqrt{5}-x^4+1}$.

15. $\frac{x}{2}-\frac{1}{8}\log(8x^2+4x+5)+\frac{1}{6}\tan^{-1}\frac{(4x+1)}{3}$.

16. $x+\log(x^2-x+1)+\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}$.

17. $-x+\log(1-x-x^2)+\frac{3}{\sqrt{5}}\log\frac{\sqrt{5}+2x+1}{\sqrt{5}-2x-1}$.

18. $\frac{x^3}{3}-\frac{x^2}{2}+\frac{1}{2}\log(x^2+x+1)-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$.

Exercises LIV (Pages 258-9)

1. $\log\frac{x+1}{x+2}$. 2. $\frac{2}{5}\log(2x+1)-\frac{11}{15}\log(1-3x)$.

3. $\log\frac{x-2}{x-1}$. 4. $\frac{1}{2}\log(x-1)-2\log(x-2)+\frac{3}{2}\log(x-3)$.

5. $\log \dfrac{(x^2-4)(x-2)}{(x+3)^2}$.

6. $\frac{1}{3}\log(x-1)+\log(x+1)-\frac{5}{6}\log(2x+1)$.

7. $\frac{23}{6}\log(2x+5)-\frac{4}{3}\log(2x-1)$.

8. $x+\log x-\frac{3}{4}\log(1-4x)$. 9. $x+\frac{3}{4}\log\dfrac{x-2}{x+2}$.

10. $\frac{2}{9}\log\dfrac{x-1}{x+2}-\frac{1}{3}\dfrac{1}{x-1}$.

11. $\frac{1}{8}\log\dfrac{x-1}{x+1}-\frac{3}{4}\dfrac{1}{x-1}-\frac{1}{4}\dfrac{1}{(x-1)^2}$.

12. $\frac{1}{9}\log\dfrac{x-1}{x+2}-\frac{2}{3}\dfrac{1}{x-1}$. 13. $3\log(x+2)+\dfrac{5}{x+2}$.

14. $\frac{1}{4}\log x-\dfrac{1}{2x}+\frac{3}{4}\log(x+2)$.

15. $\dfrac{4}{x+2}+\log(x+1)$.

16. $\frac{1}{4}\log(1+x^2)-\frac{1}{2}\log(1+x)+\frac{1}{2}\tan^{-1}x$.

17. $\frac{1}{2}\log(x+1)-\frac{1}{4}\log(x^2+1)-\dfrac{1}{2(x+1)}$.

18. $\dfrac{1}{2a^3}\left[\frac{1}{2}\log\dfrac{a+x}{a-x}+\tan^{-1}\dfrac{x}{a}\right]$.

19. $\log(x-2)-\frac{1}{2}\log(x^2+1)+\tan^{-1}x$.

20. $\frac{1}{2}\log(x^2+4)-\log(x+1)+2\tan^{-1}\dfrac{x}{2}$.

21. $\frac{1}{2}\log\dfrac{x^2+1}{x^2+3}$. 22. $\dfrac{1}{b-a}\left[\dfrac{1}{\sqrt{a}}\tan^{-1}\dfrac{x}{\sqrt{a}}-\dfrac{1}{\sqrt{b}}\tan^{-1}\dfrac{x}{\sqrt{b}}\right]$.

23. $\frac{1}{3}\log(x-1)-\frac{1}{6}\log(x^2+x+1)+\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}$.

24. $\frac{1}{6}\log(x^2-x+1)-\frac{1}{3}\log(x+1)+\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x-1}{\sqrt{3}}$.

25. $\log\dfrac{x}{\sqrt{x^2+x+1}}-\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}$.

26. $\frac{1}{2}\log(x^2+x+1)-\log x-\dfrac{1}{x}-\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x+1}{\sqrt{3}}$.

27. $\frac{1}{4\sqrt{2}}\log\frac{x^2+\sqrt{2}\,x+1}{x^2-\sqrt{2}\,x+1}+\frac{1}{2\sqrt{2}}\left\{\tan^{-1}(1-\sqrt{2}x)\right.$
$\left.-\tan^{-1}(1+\sqrt{2}x)\right\}.$

28. $\frac{1}{\sqrt{2}}[\tan^{-1}(\sqrt{2}x+1)-\tan^{-1}(1-\sqrt{2}x)].$

29. $\frac{1}{2a^2}\tan^{-1}\frac{x^2}{a^2}.$

30. $\frac{1}{12}\left[\log\frac{\sqrt{x^8-x^4+1}}{x^4+1}+\frac{4}{\sqrt{3}}\tan^{-1}\frac{2x^4-1}{\sqrt{3}}\right].$

32. $-\frac{1}{10}\log(x+1)+\frac{1}{20}\log(x^2+4x+13)$
$+\frac{11}{30}\tan^{-1}\frac{x+2}{3}.$

33. $\log\frac{1+\sin x}{2+\sin x}.$ 34. $\log\frac{\sin x}{1+\sin x}.$

35. $\frac{1}{4}\log\frac{1+2\tan x}{1-2\tan x}.$ 36. $\log\tan\frac{x}{2}+\tan\left(\frac{\pi}{4}-\frac{x}{2}\right).$

37. $\frac{1}{10}\log(\cos x-1)-\frac{1}{2}\log(\cos x+1)-4\log(3+2\cos x).$

38. $1-\log 2.$ 39. $\frac{1}{3}\log 2-\frac{\pi}{3\sqrt{3}}.$

41. $\log(\sec x+\tan x)-\cot\frac{x}{2}.$

Exercises LV (Page 262)

1. $\frac{1}{n}\log\frac{x^n}{x^n+1}.$ 2. $\frac{1}{5}\log\frac{x^5}{x^5+1}.$

3. $\frac{1}{2}\left[\log\frac{x^2}{x^2+1}+\frac{1}{x^2+1}+\frac{1}{2(x^2+1)^2}\right].$

4. $\frac{1}{7}\log\frac{x^7}{(2x^7+1)}.$ 5. $\frac{1}{2}\log\frac{x^2+1}{x^2+3}.$

6. $\frac{1}{2(a^2-b^2)}\log\frac{x^2-a^2}{x^2-b^2}.$ 7. $\frac{1}{2}\left\{\tan^{-1}x-\frac{x}{1+x^2}\right\}.$

8. $x-\frac{3}{2}\tan^{-1}x+\frac{x}{2(1+x^2)}.$

9. $-\left[\frac{1}{x}+\frac{3}{2}\tan^{-1}x+\frac{x}{2(1+x^2)}\right].$

10. $\frac{1}{2}\log(a^2+x^2)+\frac{a^2}{2(a^2+x^2)}$.

11. $\frac{1}{2}\left[\tan^{-1}(x+2)+\frac{x+2}{x^2+4x+5}\right]$.

12. $\frac{1}{2}\left[\frac{x^2(x^2+2a^2)}{x^2+a^2}-2a^2\log(x^2+a^2)\right]$.

13. $\frac{1}{2a^3b}\left[\tan^{-1}\frac{bx}{a}+\frac{abx}{a^2+b^2x^2}\right]$.

15. $\frac{1}{3}\left[\frac{1}{2}\log\frac{x-1}{x+1}+\sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}}\right]$.

16. $\frac{3}{2}\tan^{-1}x-\frac{1}{2\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}$.

Exercises LVI (Pages 266-7)

1. $\sin^{-1}(x-1)$.

2. $-2\sqrt{3+4x-x^2}+4\sin^{-1}\frac{x-2}{\sqrt{7}}$.

3. $\sqrt{2x^2-7x+5}+\frac{1}{2\sqrt{2}}\cosh^{-1}\frac{4x-7}{3}$.

4. $-4\sqrt{x^2-2x+4}-3\sinh^{-1}\frac{x-1}{\sqrt{3}}$.

5. $\sqrt{x(x-2)}+2\cosh^{-1}(x-1)$.

6. $\frac{1}{2}\sqrt{2x^2+x-3}+\frac{3}{4\sqrt{2}}\cosh^{-1}\frac{4x+1}{5}$.

7. $\sqrt{3x^2+4x+7}-\frac{6}{\sqrt{3}}\sinh^{-1}\frac{3x+2}{\sqrt{17}}$.

8. $\frac{1}{2}\sqrt{2x^2+x-3}-\frac{5}{4\sqrt{2}}\cosh^{-1}\frac{4x+1}{5}$.

9. $-\sqrt{3-5x+2x^2}+\frac{1}{2\sqrt{2}}\cosh^{-1}(4x-5)$.

10. $\sin^{-1}\frac{2x-3}{\sqrt{41}}$. 11. $2\sqrt{x^2+5x+6}-6\cosh^{-1}(2x+5)$.

12. $-2\sin^{-1}\sqrt{\frac{\beta-x}{\beta-\alpha}}$. 13. $2\cosh^{-1}x+\sqrt{x^2-1}$.

14. $\sin^{-1}\frac{2x-a+b}{a+b}$.

15. $\frac{2}{3}\sqrt{3x^2+4x+7}-\frac{16}{3\sqrt{3}}\sinh^{-1}\frac{3x+2}{\sqrt{17}}$.

16. $\sin^{-1}\dfrac{2x^2+5}{7}$. 17. $\sin^{-1}\dfrac{\sin x}{2}$. 18. $\sinh^{-1}\dfrac{\tan x}{3}$.

19. $\sqrt{x^2+x+1}+\frac{3}{2}\sinh^{-1}\dfrac{2x+1}{\sqrt{3}}$.

20. $\frac{1}{2}\left(3+11\sin^{-1}\dfrac{2}{\sqrt{5}}-11\sin^{-1}\dfrac{1}{\sqrt{5}}\right)$.

Exercises LVII (Pages 271-2)

1. $\dfrac{(x+1)}{2}\sqrt{x^2+2x+5}+2\sinh^{-1}\left(\dfrac{x+1}{2}\right)$.

2. $\frac{1}{2}(x+1)\sqrt{x^2+2x-3}-2\cosh^{-1}\left(\dfrac{x+1}{2}\right)$.

3. $\frac{1}{2}(x+1)\sqrt{3-2x-x^2}+2\sin^{-1}\left(\dfrac{x+1}{2}\right)$.

4. $-\sqrt{\dfrac{1-x}{1+x}}$. 5. $-\dfrac{(x+1)}{\sqrt{1-x^2}}$.

6. $\dfrac{2}{15}\sqrt{x+2}\,(3x^2-8x+32)$. 7. $\sqrt{\dfrac{x-1}{x+1}}$.

8. $-\dfrac{1}{\sqrt{2}}\sinh^{-1}\dfrac{1-x}{1+x}$. 9. $2\sqrt{x}-6\log(3+\sqrt{x})$.

10. $\log\dfrac{\sqrt{x+3}-1}{\sqrt{x+3}+1}$. 11. $\dfrac{(x^2+4)^{3/2}(3x^2-8)}{15}$.

12. $2\tan^{-1}\sqrt{x-1}$.

13. $\dfrac{2\sqrt{x+5}}{15}(3x^2-20x+200)$. 14. $-\dfrac{1}{\sqrt{5}}\sin^{-1}\dfrac{7-2x}{3(x-1)}$.

15. $\dfrac{2}{\sqrt{b-a}}\tan^{-1}\sqrt{\dfrac{x+a}{b-a}}$ if $b>a$.

16. $\dfrac{2\sqrt{x+1}}{15}(3x^2-4x+23)$.

17. $x-2a\sqrt{x}+2a^2\log(a+\sqrt{x})$.

18. $\log\dfrac{1+\sqrt{x}}{1-\sqrt{x}}$. 19. $-\dfrac{1}{\sqrt{109}}\sinh^{-1}\dfrac{109+3x}{10x}$.

20. $-\sinh^{-1}\dfrac{2+x}{\sqrt{3}x}$. 21. $-\dfrac{1}{\sqrt{2}}\sinh^{-1}\dfrac{\sqrt{2}}{x-1}$.

22. $\dfrac{1}{\sqrt{14}}\log\dfrac{\sqrt{2(x+5)}-\sqrt{7}}{\sqrt{2(x+5)}+\sqrt{7}}$.

23. $\dfrac{1}{\sqrt{11}}\sinh^{-1}\dfrac{3+5x}{\sqrt{2}\,(1-2x)}$.

28. $-\sinh^{-1}\dfrac{x+1}{2x}$. 29. $\dfrac{\pi}{2\sqrt{2}}$. 30. $-\dfrac{1}{\sqrt{3}}\sinh^{-1}\dfrac{2+x}{\sqrt{2}\,(x-1)}$.

Exercises LVIII (Pages 275-7)

1. $-\dfrac{\sqrt{1-x^2}}{x}$. 2. $-\dfrac{\sqrt{1+x^2}}{x}$. 3. $-\dfrac{1}{a^2}\dfrac{\sqrt{a^2-x^2}}{x}$.

4. $-\dfrac{\sqrt{a^2-x^2}}{3}(x^2+2a^2)$. 5. $\frac{1}{2}(x\sqrt{1+x^2}-\sinh^{-1}x)$.

6. $\log(x+\sqrt{1+x^2})-\dfrac{x}{\sqrt{1+x^2}}$. 7. $\dfrac{-1}{\sqrt{1+x^2}}$.

8. $\dfrac{1}{4}\left\{(2x-3)\sqrt{(x+1)(4-x)}+\dfrac{25}{2}\sin^{-1}\dfrac{2x-3}{5}\right\}$.

9. $-\sqrt{(x-1)(2-x)}-\sin^{-1}\sqrt{2-x}$.

10. $-\dfrac{1}{3a^2}\dfrac{(a^2+x^2)^{3/2}}{x^3}$.

11. $\frac{1}{2}\{\sqrt{1-x^2}(x-2+\cos^{-1}x)\}$.

12. $\sqrt{x^2-1}+\cosh^{-1}x$. 13. $-\dfrac{1}{3}\sinh^{-1}\dfrac{1}{x^3}$.

14. $\cos^{-1}\dfrac{3+x}{\sqrt{3}\,(x+1)}$. 15. $a\sin^{-1}\dfrac{x}{a}-\sqrt{a^2-x^2}$.

16. $\dfrac{2}{3a(b-c)}\{(ax+b)^{3/2}+(ax+c)^{3/2}\}$.

17. π. 18. $\pi(a+b)/2$.

21. $\dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}-\dfrac{x}{2}\sqrt{a^2-x^2}$. 22. $(x^2+8)^{1/2}+\dfrac{8}{\sqrt{x^2+8}}$.

23. $\frac{1}{3}(1-x^2)^{3/2}-2\sqrt{1-x^2}$.

24. $\dfrac{1}{2}(\sin^{-1}x-x\sqrt{1-x^2})$. 25. $\dfrac{x}{a^2\sqrt{a^2-x^2}}$.

26. $\sin^{-1}x-\sqrt{1-x^2}$. 27. $\sqrt{x(a+x)}+a\sinh^{-1}\sqrt{x/a}$.

28. $a\sin^{-1}\sqrt{x/a}+\sqrt{x(a-x)}$. 29. $-\log\tan(\pi/8)$.

30. $2\tan^{-1}(2)$. 31. $a^3(\pi-2)/4$. 32. $\pi+2$.

33. $a^2(4-\pi)/8$. 34. $-\dfrac{x}{a^2\sqrt{x^2-a^2}}$. 36. $3\pi/128$.

37. $\dfrac{1}{2}\left[\sec^{-1}x+\dfrac{\sqrt{x^2-1}}{x^2}\right]$. 38. $\dfrac{x^3}{3(1-x^2)^{3/2}}$.

39. $\dfrac{\pi(b-a)}{2}$. 40. $\sqrt{1+x^2}+\log(x+\sqrt{1+x^2})$.

41. $\sqrt{x^2-a^2}-a\sec^{-1}\dfrac{x}{a}$. 42. $\sqrt{\dfrac{1+x}{1-x}}$.

43. $2(\sqrt{e^x-1}-\tan^{-1}\sqrt{e^x-1})$. 44. $\dfrac{\pi-2}{8}$.

45. I/4 and I, where I =

$$-\sinh^{-1}x+\frac{1}{2\sqrt{2}}\log\frac{\sqrt{1+x^2}+\sqrt{2}x}{\sqrt{1+x^2}-\sqrt{2}x}.$$

Exercises LIX (Page 279)

1. $\dfrac{1}{3}\log\dfrac{\tan\frac{x}{2}+3}{3-\tan\frac{x}{2}}$. 2. $\dfrac{2}{5}\tan^{-1}\left(\dfrac{\tan x/2}{5}\right)$.

3. $\dfrac{1}{5}\log\dfrac{5+\tan\frac{x}{2}}{5-\tan\frac{x}{2}}$. 4. $\dfrac{\pi}{12}$.

5. $\dfrac{1}{3}\log\dfrac{2\tan\frac{x}{2}+1}{2\left(\tan\frac{x}{2}+2\right)}$. 6. $\dfrac{\alpha}{2\sin\alpha}$.

7. $\dfrac{1}{\sqrt{1-e^2}}\tan^{-1}\left[\sqrt{\dfrac{1-e}{1+e}}\tan\dfrac{x}{2}\right]$ if $e<1$;

$$\frac{1}{\sqrt{e^2-1}}\log\frac{\sqrt{e-1}\tan\frac{x}{2}-\sqrt{e+1}}{\sqrt{e-1}\tan\frac{x}{2}+\sqrt{e+1}} \text{ if } e>1.$$

8. $\dfrac{1}{\sqrt{7}}\log\dfrac{\sqrt{7}\tan\frac{x}{2}-1}{\sqrt{7}\tan\frac{x}{2}+1}$. 9. $\dfrac{1}{\sqrt{5}}\log\dfrac{2\tan\frac{x}{2}+3-\sqrt{5}}{2\tan\frac{x}{2}+3+\sqrt{5}}$.

10. $\dfrac{1}{2}\log\dfrac{1+\sqrt{3}\tan\frac{x}{2}}{\sqrt{3}-\tan\frac{x}{2}}$. 11 $\dfrac{1}{2\sqrt{6}}\log\dfrac{\sqrt{3}\left(\tan\frac{x}{2}-1\right)+2\sqrt{2}}{\sqrt{3}\left(-\tan\frac{x}{2}+1\right)+2\sqrt{2}}$.

12. $\log\left(1+\tan\frac{x}{2}\right)$. 14. $\frac{1}{(1-e^2)^{3/2}}(u - e\sin u)$.

15. $\sec\theta - \tan\theta - \log(\operatorname{cosec}\theta + \cot\theta)$. 16. $\pi/3$.

Exercises LX (Page 280)

1. $\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{2\sqrt{2}}\right)$. 2. $\frac{1}{\sqrt{6}}\tan^{-1}\left(\sqrt{\frac{2}{3}}\tan x\right)$.

Exercises LXI (Pages 285-6)

2. (*i*) $\pi/4$; (*ii*) $\frac{\pi}{4}(a+b)$. 3. $\pi^2/4$. 4. $\frac{a^{n+2}}{(n+1)(n+2)}$.

5. $128\sqrt{2}/105$. 6. $\frac{1}{(n+1)(n+2)}$. 7. π. 8. $\pi^2/4$.

9. $\frac{\pi}{2}(\pi-2)$. 10. $\pi/4$. 11. $(\pi/8)\log 2$. 12. 0.

13. $\pi\log 2$. 14. 0. 15. $\frac{1}{30}$. 16. $\frac{\pi^2}{4a\sqrt{a^2-1}}$.

Exercises LXII (Page 289)

1. $x(\log x - 1)$. 2. $\frac{x^4}{4}(\log x - \frac{1}{4})$.

3. $\frac{1}{2}(x^2-1)\log(x+1) - \frac{x^2}{4} + \frac{x}{2}$.

4. $x\cos^{-1}\frac{x}{a} - \sqrt{a^2-x^2}$.

5. $\frac{1}{2}x^2\sin^{-1}x - \frac{1}{4}\sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2}$.

6. $\frac{1}{2}[(x^2+1)\tan^{-1}x - x]$.

7. $\frac{x^3}{3}\sin^{-1}x + \frac{1}{9}\sqrt{1-x^2}(2+x^2)$.

8. $x\sec x - \log(\sec x + \tan x)$.

9. $\frac{1}{4}\left(x^2 - x\sin 2x - \frac{\cos 2x}{2}\right)$. 10. $\frac{e^x}{1+x}$.

11. $\frac{1}{2}[x\sqrt{a^2-x^2} + a^2\sin^{-1}x/a]$.

12. $\left(\frac{x^3}{3}-x\right)\tan^{-1}x + \frac{2}{3}\log(1+x^2) + \frac{(\tan^{-1}x)^2}{2} - \frac{x^2}{6}$.

13. $e^x\tan\frac{x}{2}$. 14. $\frac{1}{2}$. 15. $x - \sqrt{1-x^2}\sin^{-1}x$.

16. $\frac{\pi}{4} - \frac{1}{2}\log 2$. 17. $\frac{\pi-1}{4}$.

It is natural to call a function continuous if its graph is continuous or smooth and otherwise discontinuous. Taking this as a provisional definition of continuous functions, we shall try to get some of the properties of those functions. To define continuity for all values of x, we must first define continuity for one particular value of x, say, the value of $x = a$, at the point P of the graph A in Fig. 2. We shall find the characteristic properties of $f(x)$ associated with the value of x.

1. $f(x)$ must be defined for $x = a$. This is essential for if $f(a)$ were not defined, there would be a point missing from the curve.

2. $f(x)$ is defined for all values near $x = a$.

3. When x approaches the value a from either side, $f(x)$ approaches the limit $f(a)$

i.e., $\underset{x \to a}{\text{Lt}} f(x) = f(a)$.

Selecting the above three properties as embodying the notion of mathematical continuity at a point, we are led to the following definition :—

The function $f(x)$ is said to be continuous at $x = a$ if it tends to a limit as $x \to a$ from either side and each of these limits is equal to $f(a)$.

We can now define continuity in an interval. A function is said to be continuous in the interval (a, b) if it is continuous at every point in the interval.

We can easily see that any polynomial is continuous for all values of x and any rational fraction is continuous except for values of x for which the denominator vanishes, *i.e.*, $R(x) = \dfrac{P(x)}{Q(x)}$ is discontinuous for $x = a$, where a is any root of $Q(x) = 0$. Thus $\dfrac{2x + 7}{x^2 - 4x + 3}$ is discontinuous for $x = 1$ and $x = 3$.

Sin x, cos x are continuous functions for all values of x. Tan x and sec x are continuous functions for x except for values of x equal to an odd multiple of $\frac{\pi}{2}$. Similarly cosec x is a continuous function except when x is a multiple of π.

17. $\underset{x \to \infty}{\text{Lt}} \left(\sqrt{x^2 + x + 1} - x\right)$.

18. $\underset{n \to \infty}{\text{Lt}} \left(\frac{1}{3} + \frac{1}{9} + \ldots\ldots + \frac{1}{3^n}\right)$.

19. $\underset{n \to \infty}{\text{Lt}} \left(\frac{1^2 + 2^2 + \ldots\ldots + n^2}{n^3}\right)$.

20. $\underset{\theta \to 0}{\text{Lt}} \frac{\sin a\theta}{\sin b\theta}$.

21. $\underset{\theta \to 0}{\text{Lt}} \frac{\tan a\theta}{\tan b\theta}$.

22. $\underset{\theta \to 0}{\text{Lt}} \frac{\sin m\theta}{\tan n\theta}$.

23. $\underset{\theta \to 0}{\text{Lt}} \frac{1 - \cos m\theta}{1 - \cos n\theta}$.

24. $\underset{x \to \frac{\pi}{2}}{\text{Lt}} \frac{\cot x}{\frac{\pi}{2} - x}$.

25. $\underset{x \to 0}{\text{Lt}} \frac{\sin^{-1} x}{x}$.

26. $\underset{x \to 0}{\text{Lt}} \frac{\operatorname{cosec} x - \cot x}{x}$.

27. $\underset{x \to 1}{\text{Lt}} \frac{1 + \cos \pi x}{\tan^2 \pi x}$.

28. $\underset{x \to \frac{\pi}{2}}{\text{Lt}} (\sec x - \tan x)$.

29. $\underset{x \to 1}{\text{Lt}} \frac{x^{-1/3} - 1}{x^{-2/3} - 1}$.

§ 11. Continuous and discontinuous functions.

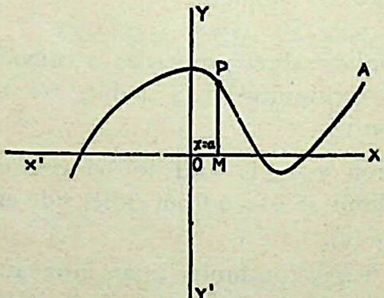

Fig. 2

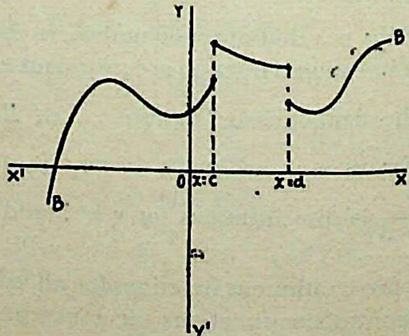

Fig. 3

The curve in Fig. 2 is continuous and the curve in Fig. 3 is generally continuous but discontinuous for $x = c$ and $x = d$.

c—2

Ex. 3. Show that $\underset{x\to 0}{\text{Lt}} \dfrac{(1+x)^n - 1}{x} = n.$

Put $(1+x)^n - 1 = y.$

$\therefore \quad (1+x)^n = (1+y).$

Taking logarithms of both sides, we get

$n \log(1+x) = \log(1+y).$

$$\underset{x\to 0}{\text{Lt}} \frac{(1+x)^n - 1}{x} = \underset{x\to 0}{\text{Lt}} \left\{ \frac{(1+x)^n - 1}{\log(1+x)} \cdot \frac{\log(1+x)}{x} \right\}$$

$$= \underset{y\to 0}{\text{Lt}} \frac{y}{\frac{1}{n}\log(1+y)} \cdot \underset{x\to 0}{\text{Lt}} \log(1+x)^{1/x}$$

$$= \underset{y\to 0}{\text{Lt}}\, n \cdot \frac{1}{\log(1+y)^{1/y}} \underset{x\to 0}{\text{Lt}} \log(1+x)^{1/x}$$

$$= \frac{n}{\log e} \log e$$

$$= n.$$

We can easily find this limit by using § 10 (1).

Exercises II.

Evaluate the following :—

1. $\underset{x\to 0}{\text{Lt}} \dfrac{cx+d}{ax+b}.$

2. $\underset{x\to 0}{\text{Lt}} \dfrac{2x^2-4x+1}{4x^2+6x+5}.$

3. $\underset{x\to 2}{\text{Lt}} \dfrac{x^3-3x^2+2}{x^2-2x-5}.$

4. $\underset{x\to 1}{\text{Lt}} \dfrac{x^4-1}{x-1}.$

5. $\underset{x\to 2}{\text{Lt}} \dfrac{x^3-8}{x-2}.$ 16

6. $\underset{x\to 2}{\text{Lt}} \dfrac{x^2-5x+6}{x^2-4}.$

7. $\underset{x\to 1}{\text{Lt}} \dfrac{x^4-3x^3+2}{x^3-5x^2+3x+1}.$

8. $\underset{x\to 0}{\text{Lt}} \dfrac{(1+x)^4-1}{(1+x)^2-1}.$

9. $\underset{x\to 0}{\text{Lt}} \dfrac{(1+x)^3-1-3x}{(1+x)^2-1-2x}.$

10. $\underset{x\to\infty}{\text{Lt}} \dfrac{4x+8}{3x-7}.$

11. $\underset{x\to\infty}{\text{Lt}} \dfrac{x^2+b^2}{x^3+a^3}.$

12. $\underset{x\to\infty}{\text{Lt}} \dfrac{(3x+1)(2x+4)}{(x+1)(x-7)}.$

13. $\underset{x\to 0}{\text{Lt}} \dfrac{\sqrt{1+x}-1}{x}.$

14. $\underset{x\to a}{\text{Lt}} \dfrac{\sqrt{3a-x}-\sqrt{x+a}}{x-a}.$

15. $\underset{x\to 0}{\text{Lt}} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}.$

16. $\underset{x\to 2}{\text{Lt}} \dfrac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}.$

As m tends to infinity, $(m-1)$ also tends to infinity.

$$\therefore \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^n$$

$$= \underset{m-1\to\infty}{\text{Lt}} \left(1+\frac{1}{m-1}\right)^{m-1} \underset{m-1\to\infty}{\text{Lt}} \left(1+\frac{1}{m-1}\right)$$

$$= e \qquad \text{[by cases } (i) \text{ and } (ii)].$$

$$\therefore \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^n = e \text{ for all rational values of } n.$$

We can easily see that $\left(1+\frac{1}{n}\right)^n$ is greater than 2 from its expansion.

$$\therefore \quad 2 < e < 3.$$

Again e is an incommensurable number, *i.e.*, e cannot be a fraction of the form $\frac{a}{b}$. The value of e has been calculated to more than 500 decimal places. The value of e is

$$2{\cdot}7182818285 \ldots\ldots$$

Examples.

Ex. 1. Find $\underset{x\to 0}{\text{Lt}} (1+x)^{1/x}$.

Substituting $\frac{1}{n}$ for x, we get

$$\underset{x\to 0}{\text{Lt}} (1+x)^{1/x} = \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^n$$

$$= e.$$

Ex. 2. Find $\underset{x\to 0}{\text{Lt}} \dfrac{e^x-1}{x}$.

Put $e^x - 1 = y \quad \therefore \quad y+1 = e^x$.

$$\therefore \quad x = \log_e (y+1).$$

As $x \to 0$, $y \to 0$.

$$\therefore \underset{x\to 0}{\text{Lt}} \frac{e^x-1}{x} = \underset{y\to 0}{\text{Lt}} \frac{y}{\log_e (y+1)}$$

$$= \underset{y\to 0}{\text{Lt}} \frac{1}{\log_e (y+1)^{1/y}}$$

$$= \frac{1}{\log_e e}$$

$$= 1.$$

Exercises XCIII (Page 409)

$y = A\cos(x + B) + \frac{e^x}{2}(x^2 - 2x + 3/2).$

$y = A\cos(2x + B) + \frac{e^{2x}}{8}(x - \frac{1}{2}).$

$y = A e^{3x} + B e^x - e^x \frac{\sin 2x + \cos 2x}{8}.$

Basanna

$y = e^x \{ A\cos x + (B + x/2)\sin x \}.$

$y = e^x A\cos(\sqrt{3}x + B) + \frac{\sin x}{2}.$

Basanna

CHAPTER XVI

Exercises XCIV (Page 420)

Prasanna

1. $y = cx + \frac{c}{(1 + c^2)^{1/2}}.$
2. $(y - 2x - c)(y - 3x - c) = 0.$
3. $x + c = \log(p + \sqrt{1 + p^2})$; $y = \sqrt{1 + p^2}.$
4. $y - c = \sqrt{x^2 - c^2} - \tan^{-1}\sqrt{\frac{1 - x}{x}}.$
5. $y = cx + \frac{a}{c}.$
6. $y^2 = x^2 c + c^2.$
7. $(xy - c)(y^2 - x^2 - 2c) = 0.$
8. $e^y = ce^x + c^3.$
9. $xy = ce^{-y/x}.$
10. $y = 2c\sqrt{x} + c^4.$
11. $[y(1 + \cos x) - c][y(1 - \cos x) - c] = 0.$
12. $\left. \begin{aligned} x &= -\frac{ap}{\sqrt{p^2 - 1}}\cosh^{-1} p + c; \\ y &= 2px + x^2p^4. \end{aligned} \right\}$
13. $(y - x - c)(x^2 + y^2 - 2c) = 0.$
14. $y^3 = axy + cx^2.$
15. $y^2 = x^2 c - \frac{2c}{1 + c}.$

Exercises XCV (Page 426)

1. $y = Ax^{\sqrt{3}} + Bx^{-\sqrt{3}} + x^2.$
2. $y = Ax^{-3} + Bx^{-4} + \frac{x^4}{56}.$

3. $y = A + Bx^{-1} + 3x^2 + 2x.$

4. $y = Ax^{\sqrt{2}} + Bx^{-\sqrt{2}} + \dfrac{x^2}{6}.$

5. $y = x^2 (A + B \log x) + \dfrac{x^2}{2} (\log x)^2.$

6. $y = Ax^{-1} + Bx^5 + \dfrac{2 \cos \log x - 3 \sin \log x}{26}.$

7. $y = x^{-3} \{ A \cos (2 \log x) + B \sin (2 \log x) \} + \dfrac{1}{13} \Big(\log x$

8. $y = x^{5/2} \left\{ Ax^{\sqrt{17}/2} + Bx^{-\sqrt{17}/2} \right\} + \dfrac{1}{2} \left(\log x + \dfrac{5}{2} \right).$

9. $y = \dfrac{Ax^6}{120} + \dfrac{Bx^5}{60} + \dfrac{x^3}{9} + \dfrac{Cx^2}{2} + Dx + E.$

10. $y = A \cos \log x + B \sin \log x + \log x + \dfrac{x}{2}.$

Exercises XCVI (Page 427)

1. $y = A (x + a)^2 + B (x + a)^3 + \dfrac{3x + 2a}{6}.$

2. $y = (x + 1)^2 \{ A + B \log (x + 1) \} + \dfrac{1}{4} \Big(x^2 + 3x + \dfrac{7}{2} \Big)$

3. $y = A (x + a)^2 + B (x + a)^3 + \dfrac{1}{6} \left(x + \dfrac{5}{6} \right).$

4. $$y = (1 + 2x)^{-1/4} \left\{ A \cos \left[\frac{\sqrt{3}}{4} \log (1 + 2x) \right] + B \sin \left[\frac{\sqrt{3}}{4} \log (1 + 2x) \right] \right\} + 8 (1 + 2x)^2 - 32 (1 + 2x)$$